1. (14 points) Evaluate the following definite and indefinite integrals. If necessary, use substitution. Show all of your work.

(a) \[ \int \frac{5x^4 + 2x^2}{x^3} + \pi \, dx = \int \left( \frac{5x}{x^3} + \frac{2x^2}{x^3} + \pi \right) \, dx = \int (5x + 2x^{-1} + \pi) \, dx = \frac{5}{2} x^2 + 2 \ln |x| + \pi x + C \]

(b) \[ \int t^3 \sin(t^4 + 7) \, dt = \int \frac{1}{4} \sin(\omega) \, d\omega = -\frac{1}{4} \cos(\omega) + C = -\frac{1}{4} \cos(t^4 + 7) + C \]

\[ \omega = t^4 + 7 \]

\[ d\omega = 4t^3 \, dt \]

\[ \frac{1}{4} \, d\omega = t^3 \, dt \]

(c) \[ \int_0^1 \frac{3}{(2x + 1)^3} \, dx = \int_1^3 \frac{3}{2} \frac{1}{\omega^3} \, d\omega = \frac{3}{2} \left[ \frac{1}{\omega^{-2}} \right]_1^3 = -\frac{3}{4} \frac{1}{\omega^2} \left|_1^3 \right. = -\frac{3}{4} \left( \frac{1}{9} - \frac{1}{1} \right) = -\frac{3}{9} \left( 9 \right) = \frac{3}{3} \]

(d) Use integration by parts to evaluate \[ \int 5xe^{4x} \, dx \].

Let \[ u = 5x \], \[ dv = e^{4x} \, dx \]

\[ du = 5 \, dx \], \[ v = \frac{1}{4} e^{4x} \]

\[ \int 5xe^{4x} \, dx = uv - \int v \, du = 5x \left( \frac{1}{4} e^{4x} \right) - \int \frac{1}{4} e^{4x} \cdot (5 \, dx) = \frac{5}{4} xe^{4x} - \frac{5}{4} \left( \frac{1}{4} e^{4x} \right) + C = \frac{5}{4} xe^{4x} - \frac{5}{16} e^{4x} + C \]
2. (14 points) Consider the discrete-time dynamical system

\[ N_{t+1} = 2N_t(1 - N_t) - hN_t \]

describing a population of shrimp being harvested at rate \( h \geq 0 \).

(a) Find the nonzero equilibrium population \( N^* \) as a function of \( h \). For what values of \( h \) is there a positive equilibrium?

\[
\begin{align*}
N^* &= 2N^*(1 - N^*) - hN^* \\
N^* &= 2N^* - 2N^2 - hN^* \\
0 &= N^* - 2N^2 - hN^* \\
&= N^*(1 - h - 2N^*) \\
\Rightarrow N^* &= \frac{1 - h}{2} \text{ if } h < 1 \\
\end{align*}
\]

(b) The equilibrium harvest is given by \( P(h) = hN^* \), where \( N^* \) is the equilibrium you found in part (a). Find the value of \( h \) that maximizes \( P(h) \). Use the first or second derivative test to justify that this value of \( h \) gives a local maximum.

\[
P(h) = hN^* = h\left(\frac{1 - h}{2}\right) = \frac{1}{2}(h - h^2)
\]

\[
P'(h) = \frac{1}{2} - h
\]

Critical point: \( P'(h) = 0 \Rightarrow h = \frac{1}{2} \).

1st derivative test:

\[
\begin{array}{c|c|c|c}
\text{h} & P'(h) & \text{sign of } P'(h) & \text{behavior} \\
\hline
< \frac{1}{2} & + & - & \text{decreasing} \\
\frac{1}{2} & 0 & & \text{local max} \\
\frac{1}{2} & - & + & \text{increasing} \\
\end{array}
\]

2nd derivative test:

\[
P''(h) = \frac{1}{2}
\]

\[
\Rightarrow P''(\frac{1}{2}) = \frac{1}{2} > 0 \Rightarrow P \text{ has a local max at } h = \frac{1}{2}.
\]

(c) Use the Stability Test/Criterion to determine if the equilibrium you found in (a) is stable if \( h = \frac{1}{2} \).

\[
\begin{align*}
N^* &= 2N^*(1 - N^*) - hN_t = (2 - h)N^* - 2N^2 = P(N^*_t) \\
P'(N^*_t) &= 2 - h - 4N^*_t
\end{align*}
\]

\[
N^* = \frac{1 - h}{2} \text{ is stable if } |P'(N^*_t)| < 1
\]

\[
|P'(\frac{1}{2})| = |2 - h - 4(\frac{1}{2})| = |2 - h - 2(1 - h)| = |h|, \text{ which is } < 1 \text{ if } h < 1
\]

\[
-1 < h < 1
\]

\[
( h \geq 0, \text{ so actually } 0 \leq h < 1
\]

\[
\Rightarrow N^* = \frac{1 - h}{2} \text{ is stable if } 0 \leq h < 1
\]
3. (15 points) Consider the function \( f(x) = -x^3 + 3x^2 + 9x \) on the interval \([-3.5, 4]\).

(a) Calculate \( f'(x) \), and use this to find all the critical points of \( f(x) \).

\[
\begin{align*}
f'(x) &= -3x^2 + 6x + 9 \\
\Rightarrow x^2 - 2x - 3 &= 0 \\
(x+1)(x-3) &= 0 \\
\Rightarrow \text{critical points:} \\
x &= -1 \\
x &= 3
\end{align*}
\]

(b) Calculate \( f''(x) \), and use this to find regions where \( f(x) \) is concave up or concave down.

\[
\begin{align*}
f''(x) &= -6x + 6 \\
-6x + 6 &= 0 \\
\Rightarrow x &= 1
\end{align*}
\]

\[
\begin{align*}
\text{If } f''(x) > 0 \text{ for } x < 1: & \quad \text{concave up for } x < 1 \\
\text{if } f''(x) < 0 \text{ for } x > 1: & \quad \text{concave down for } x > 1.
\end{align*}
\]

(c) For each critical point, determine if \( f(x) \) has a local maximum or a local minimum there. Justify your answer using the first derivative test.

\( x = -1 \):

\[
\begin{align*}
f'(-2) &= -15 < 0 \\
f'(0) &= 9 > 0 \\
f'(3) &= 0 \\
f'(4) &= -17 < 0
\end{align*}
\]

\( f \) has a local minimum at \( x = -1 \).

\( x = 3 \):

\( f \) has a local maximum at \( x = 3 \).

(d) Use the information found above to sketch a graph of the function \( f(x) \) on the interval \([-3.5, 4]\). Indicate where any local maxima, local minima, global maxima, or global minima occur.
4. (14 points) Let \( P(t) \) be the amount (in moles) of a chemical being formed in a reaction. Suppose that the rate at which the chemical is being formed is given by

\[
\frac{dP}{dt} = \frac{2t}{t^2 + 5} \text{ moles/sec.}
\]

(a) What is \( \lim_{t \to \infty} \frac{dP}{dt} \)?

\[
\lim_{t \to \infty} \frac{2t}{t^2 + 5} = \lim_{t \to \infty} \frac{2t}{t^2} = \lim_{t \to \infty} \frac{2}{t} = 0.
\]

(b) If \( P(0) = 5 \), what is \( P(t) \)? (Solve the initial-value problem for \( P(t) \)).

\[
P(t) = \int \frac{2t}{t^2 + 5} \, dt = \int \frac{1}{w} \, dw = \ln|w| + C
\]

\[
= \int \frac{2t}{t^2 + 5} \, dt = \int \frac{2t}{t^2 + 5} \, dt = \ln|t^2 + 5| + C
\]

\[
= \ln|t^2 + 5| + C
\]

\[
= \ln|t^2 + 5| + (5 - \ln(5))
\]

(c) Find the average rate at which product is being formed (that is, the average value of \( \frac{dP}{dt} = \frac{2t}{t^2 + 5} \) between times \( t = 0 \) and \( t = 2 \)).

\[
\text{Average rate of change} = \frac{1}{2 - 0} \int_0^2 \frac{2t}{t^2 + 5} \, dt = \frac{1}{2} \cdot \ln|t^2 + 5| \bigg|_0^2
\]

\[
= \frac{1}{2} \ln(9) - \frac{1}{2} \ln(5)
\]

\[
= \frac{1}{2} \ln \left( \frac{9}{5} \right) = 0.2938
\]

(d) Use a definite integral to find the total change in the amount of product between times \( t = 1 \) and \( t = 5 \).

\[
\Delta P \text{ between } t = 1 \text{ and } t = 5 = \int_1^5 \frac{2t}{t^2 + 5} \, dt = \int_1^5 \frac{2t}{t^2 + 5} \, dt = \ln|t^2 + 5| \bigg|_1^5
\]

\[
= \ln(5^2 + 5) - \ln(1 + 5) = \ln \left( \frac{30}{6} \right) = \ln(5)
\]

\[
\approx 1.6094
\]
5. (14 points) Suppose that a cell is absorbing a certain drug from its environment. At time 
t = 0, there is 10 mol of the drug in the cell, and the drug enters the cell at a rate of 1 + \sin(t^2) 
mol/min.

(a) Let \( c(t) \) represent the amount (mol) of drug in the cell at time \( t \) (in minutes). Write a 
pure-time differential equation and an initial condition for the situation described above.

\[
\frac{dc}{dt} = 1 + \sin(t^2) \\
\hspace{1cm} c(0) = 10
\]

(b) Apply Euler's Method with \( \Delta t = 0.5 \) to estimate the amount of drug in the cell at time 
\( t = 1.5 \). Show your work clearly using a table.

(Recall the formula \( \hat{c}_{\text{next}} = \hat{c}_{\text{current}} + \frac{dc}{dt} \Delta t \), or \( \hat{c}(t + \Delta t) = \hat{c}(t) + c'(t) \Delta t. \))

<table>
<thead>
<tr>
<th>( t )</th>
<th>( \hat{c}_{\text{current}} )</th>
<th>( \frac{dc}{dt} = 1 + \sin(t^2) )</th>
<th>( \hat{c}<em>{\text{next}} = \hat{c}</em>{\text{current}} + 0.5 \left( 1 + \sin(2.25) \right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>10</td>
<td>1</td>
<td>10 + 0.5 \cdot 1 = 10.5</td>
</tr>
<tr>
<td>0.5</td>
<td>10.5</td>
<td>1.2474</td>
<td>10.5 + 0.5(1.2474) = 11.1237</td>
</tr>
<tr>
<td>1</td>
<td>11.1237</td>
<td>1.8415</td>
<td>11.1237 + 0.5(1.8415) = 12.04445</td>
</tr>
<tr>
<td>1.5</td>
<td>12.04445</td>
<td></td>
<td>( \hat{c}(1.5) = 12.044 )</td>
</tr>
</tbody>
</table>

\( \hat{c}(1.5) = 12.044 \)
6. (14 points)

(a) A flying big brown bat slows down a bit to catch a fly, and then increases its speed again as it flies on. Denoting the position (in meters) of the bat at time \( t \) (in seconds) by \( P(t) \), suppose that the bat’s velocity is given by

\[
\frac{dP}{dt} = 1 - \frac{t^2}{1 + t^4}.
\]

Estimate the total change in \( P(t) \) between times \( t = 0 \) and \( t = 2 \) using a right-hand Riemann Sum with \( \Delta t = 0.5 \). Draw your rectangles or step functions on the graph below.

\[
\Delta P \text{ between } t = 0 \text{ and } t = 2 \approx \Delta t \left( \frac{dP}{dt}(0.5) + \frac{dP}{dt}(1) + \frac{dP}{dt}(1.5) + \frac{dP}{dt}(2.0) \right) = 0.5 \left( 1 - \frac{0.5^2}{1+0.5^4} + 1 - \frac{1^2}{1+1^4} + 1 - \frac{1.5^2}{1+1.5^4} + 1 - \frac{2^2}{1+2^4} \right) \approx 1.3291
\]

(b) The density of a very thin rod varies according to \( \rho(x) = \frac{1}{10}xe^{-x^2} \) in grams/cm, where \( x \) marks a location along the rod and \( x = 0 \) at one end of the rod. What is the total mass of the rod if it is 2 cm long?

\[
\text{Total mass} = \int_0^2 \rho(x) \, dx = \int_0^2 \frac{1}{10}xe^{-x^2} \, dx
\]

\[
\begin{align*}
\int \frac{1}{10} \, dx & = \frac{1}{10} \left[ x - \frac{1}{2}e^{-x^2} \right]_0^2 \\
& = \frac{1}{10} \left( 2 - \frac{1}{2}e^{-4} \right) \\
& \approx \frac{1}{10} \left( 2 - 0.067 \right) \\
& \approx 0.173 \\
\int -2x \, dx & = -x^2 + C \\
\int 2x \, dx & = x^2 + C \\
\int e^{-x^2} \, dx & = \frac{1}{2} \sqrt{\pi} \text{erf}(x) + C \\
\int dx & = x + C
\end{align*}
\]

\[
\text{Total mass} = 0.173 + 0.067 = 0.24 \text{ grams}
\]
7. (15 points)

(a) A population of green, slimy algae obeys the discrete-time dynamical system

\[ a_{t+1} = 1.8a_t. \]

(i) Write down the solution of this discrete-time dynamical system if \( a_0 = 500. \)

(ii) If \( a_0 = 500, \) at what time does the population reach size 1000?

\[ \begin{align*}
(\text{i}) & \quad a_t = 500 \cdot 1.8^t \\
(\text{ii}) & \quad 1000 \leq 500 \cdot 1.8^t \\
& \quad 2 = 1.8^t \\
& \quad \ln(2) = t \cdot \ln(1.8) \\
& \quad t = \frac{\ln(2)}{\ln(1.8)} \approx 1.79
\end{align*} \]

(b) Use a tangent-line approximation of the function \( f(x) = \sqrt{x} \) to approximate \( \sqrt{3.9}. \) Give your answer to 3 decimal places.

\[ \begin{align*}
& \text{Tangent-line approximation of } f(x) = \sqrt{x} \text{ at } x = 4 : \\
& \quad f(x) - f(4) = f'(4)(x-4); \quad (f'(x) = \frac{1}{2}x^{-\frac{1}{2}}) \\
& \quad \hat{f}(x) = f(4) + \frac{1}{2}(x-4) = \frac{1}{2}(x-4); \\
& \quad \text{Approximation of } \sqrt{3.9} = \hat{f}(3.9) = \frac{1}{2}(3.9-4) + 2 = -0.025 + 2 = 1.975
\end{align*} \]

(c) Suppose that the population \( k(t) \) of green kingfishers (measured in thousands) satisfies the differential equation

\[ \frac{dk}{dt} = 3.1k(2.2 - k). \]

i) Find all equilibria of the differential equation.

\[ 0 = \frac{3.1k(2.2 - k)}{k} \Rightarrow \begin{cases} 
\text{ke} = 0, 0 \text{ or } \\
\text{ke} = 2.2
\end{cases} \]

ii) Write down an initial condition for which the population will increase with time.

\[ k(0) = 1.69, (0, 2.2). \]

\[ \text{Any ke}(t) \text{ such that } 0 < \text{ke}(t) < 2.2 \text{ at } \text{ms.} \]