Math155: Calculus for Biological Scientists
A model of the heart

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In this lecture we will study the dynamics of heartbeats.

Heartbeats are regulated by two collections of cells in the heart:
1. the **SA node** (sinoatrial node)
2. the **AV node** (atrioventricular node).
A model of the heart

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Picture by J. Heuser.
A model of the heart

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Whether conditions are suitable for a heartbeat is determined by the electric potential of the AV node: if the potential is too high, there is no heartbeat. Electric potential is measured in volts.

If the electric potential is however below a certain threshold value, denoted $W$, then the AV node tells the heart to beat and increases the potential by a certain amount, denoted $u$. We will assume $W > 0$.

Note: $W$ is called $V_c$ in the book.
A model of the heart

Let $V_t$ denote the potential in the AV node just after a the AV node has responded to a signal from the SA node. Let $\tau$ denote the time between two signals from the SA node. Then $\tau$ is also the time interval between measurements of the potential $V_t$. 
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This means that the potential just before the next signal from the SA node is

$$e^{-\alpha \tau} V_t$$

Let $c = e^{-\alpha \tau}$ and note that $0 < c < 1$. Then $e^{-\alpha \tau} V_t = cV_t$. 
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The potential just before the next signal from the SA node is equal to $cV_t$.

Therefore:

$$V_{t+1} = cV_t \quad \text{if} \quad cV_t > W$$
$$V_{t+1} = cV_t + u \quad \text{if} \quad cV_t \leq W$$
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• If \( cV_t > W \) then the potential is too high and there is no heartbeat. The potential value \( cV_t \) is not increased by \( u \).

• But if \( cV_t \leq W \) then there is a heartbeat and the potential is increased by \( u \).
A model of the heart

The updating function of the dynamical system of the heart is

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For example, we could have \( c = 0.4 \), \( u = 1 \) and \( W = 1 \).

For these values of the parameters, let us graph the updating function. Before drawing the graph we will take a look at the updating function. First we plug in \( c = 0.4 \) and \( u = W = 1 \):

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V_{t+1} &= 0.4V_t \quad \text{if} \quad 0.4V_t > 1 \\
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- \( 0.4V_t > 1 \) becomes \( V_t > 1/0.4 \)
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- \(0.4V_t \leq 1\) becomes \(V_t \leq 1/0.4\)

Since \(1/0.4 = 2.5\) we get:

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The graph of the updating function and the diagonal

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An equilibrium

In this case there is an equilibrium (about 1.7).

\[ u = 1, \quad c = 0.4, \quad W = 1 \]

**updating function**

**diagonal**
More on equilibria

At an equilibrium the heart will beat steadily and every signal sent by the SA node results in a heartbeat.
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For example, if $c = 0.6$, $u = 1$ and $W = 1$, then there is no equilibrium.
No equilibria

If $c = 0.6$ and $u = W = 1$ then there is no equilibrium.
Computing equilibria

\[ V_{t+1} = cV_t \quad \text{if} \quad cV_t > W \]
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An equilibrium \( V^*_t \) must satisfy \( cV^*_t \leq W \) since otherwise the heart would not beat after the potential has fallen from \( V^*_t \) to \( cV^*_t \). Therefore, that \( V^*_t \) is an equilibrium means that \( V^*_t = cV^*_t + u \) and \( cV^*_t \leq W \).

If there exists a solution, then it is given by \( V^*_t = u \) and \( cV^*_t \leq W \).
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If there exists a solution, then it is given by

\[ V^* = \frac{u}{1 - c} \quad \text{and} \quad cV^* \leq W \]
Example (A heart beating with every signal)

Consider again the case $c = 0.4$, $u = 1$ and $W = 1$. The graph from above indicated that there is an equilibrium in this case.
A steady heartbeat

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$$0.4 \cdot (1/0.6) \leq 1$$

But $0.4 \cdot (1/0.6) = 0.4/0.6 = 2/3$ and hence the inequality is fulfilled!
A steady heartbeat

Example (A heart beating with every signal)

We conclude that there really is an equilibrium $V^* = 1/0.6$. 

Note that $V^* = 1/0.6 = 10/6 = 5/3$. 

What happens in this example if we are at the equilibrium potential? Let us assume that $V_t = V^* = 5/3$. The AV node has just responded to a signal from the SA node. Recall that $c = 0.4$. The next event is that the potential drops to $0.4V^* = 0.4 \cdot (10/6) = (0.4 \cdot 10)/6 = 4/6 = 2/3$. 

When the next signal arrives from the SA node, the AV node checks whether the potential ($2/3$ volts) has dropped below the threshold $W$ (1 volt). It has, and therefore the heart beats.
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Next, the potential is increased by $u$ (1 volt) to

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We are back exactly at the equilibrium potential $V^*$ again!
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The procedure will start over and the heart beats steadily. This is what it means to be at an equilibrium.
Not a steady heartbeat

Example (A heart failing to beat with some signals)

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Next, the potential falls to $0.6 \cdot 1.5 = 0.9$ volts until the next signal from the SA node. Now the potential is smaller than 1 and the heart beats! After the heartbeat the potential is raised by 1.0 volts.
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We see that in this case the heart does not beat with every signal from the SA node.
On existence of equilibria

<table>
<thead>
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Recall that $c = e^{-\alpha \tau}$ where $\tau$ is the time between signals from the SA node and $0 < \alpha$ is a constant. As $\tau$ gets smaller, $c$ gets bigger. This can lead to heart beats being skipped. That is, if the signals from the SA node come too close to-gether, it is trying to make the heart beat too fast and some beats may be skipped.
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In the first example the heart beats with every signal from the SA node (there was an equilibrium) and in the second example this was not the case.
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AV block

We will now study a condition called **2:1 AV block**. Under this condition, the heart does not beat with every signal, in fact it beats only with every other signal.
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Here is a diagram illustrating what happens during 2:1 AV block:

\[ V_t \xrightarrow{\text{decay}} cV_t \xrightarrow{\text{signal ignored}} cV_t \xrightarrow{\text{decay}} c^2V_t \xrightarrow{\text{signal obeyed}} c^2V_t + u \]
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The signal is obeyed after two cycles and then the potential should be equal to \(V_t\) again:

\[V_t = c^2V_t + u\]
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Let $\overline{V}$ denote a potential where 2:1 AV block arises. Here is the diagram again:

\[ \overline{V} \xrightarrow{\text{decay}} c\overline{V} \xrightarrow{\text{signal ignored}} c\overline{V} \xrightarrow{\text{decay}} c^2\overline{V} \xrightarrow{\text{signal obeyed}} c^2\overline{V} + u \]
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$\overline{V} \xrightarrow{\text{decay}} c\overline{V} \xrightarrow{\text{signal ignored}} c\overline{V} \xrightarrow{\text{decay}} c^2\overline{V} \xrightarrow{\text{signal obeyed}} c^2\overline{V} + u$

Since the first signal is ignored we have that $c\overline{V} > W$. But the second signal is obeyed and therefore $c^2\overline{V} \leq W$. 
AV block

Let $\overline{V}$ denote a potential where 2:1 AV block arises. Here is the diagram again:

$\overline{V}$ decay $\rightarrow$ $c\overline{V}$ signal ignored $\rightarrow$ $c\overline{V}$ decay $\rightarrow$ $c^2\overline{V}$ signal obeyed $\rightarrow$ $c^2\overline{V} + u$

Since the first signal is ignored we have that $c\overline{V} > W$. But the second signal is obeyed and therefore $c^2\overline{V} \leq W$.

The conditions on $\overline{V}$ are:

$\overline{V} = c^2\overline{V} + u$  \hspace{1cm} $c\overline{V} > W$  \hspace{1cm} $c^2\overline{V} \leq W$
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\[ \overline{V} = \frac{u}{1 - c^2} \quad c\overline{V} > W \quad c^2\overline{V} \leq W \]
Let $\overline{V}$ denote a potential where 2:1 AV block arises. Here is the diagram again:

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\overline{V} \xrightarrow{\text{decay}} c\overline{V} \quad \text{signal ignored} \quad \overline{V} \xrightarrow{\text{decay}} c^2\overline{V} \quad \text{signal obeyed} \quad c^2\overline{V} + u
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\overline{V} = \frac{u}{1 - c^2} \quad c\overline{V} > W \quad c^2\overline{V} \leq W
$$

The inequalities may or may not be fulfilled, depending on the parameters $c$, $u$ and $W$. 
An example of 2:1 AV block

Now let us look at a particular example where 2:1 AV block occurs: \( c = \frac{2}{3} \) and \( u = W = 1 \).
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For \( \bar{V} \) we get

\[
\bar{V} = \frac{u}{1 - c^2} = \frac{1}{1 - (\frac{2}{3})^2} = \frac{1}{1 - \frac{4}{9}} = \frac{9}{9 - 4} = \frac{9}{5} = 1.8
\]
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We have to check the inequalities \( c\overline{V} > W \) and \( c^2\overline{V} \leq W \):

\[
\frac{2}{3} \cdot \frac{9}{5} = \frac{6}{5} = 1.2 > 1 \quad \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{9}{5} = \frac{4}{5} = 0.8 < 1
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\]

Here is the diagram from above:

\[
1.8 \xrightarrow{\text{decay}} c \cdot 1.8 = 1.2 \xrightarrow{\text{signal ignored}} 1.2 \xrightarrow{\text{decay}} c \cdot 1.2 = 0.8
\]

\[
\text{signal obeyed} \rightarrow 0.8 + u = 0.8 + 1 = 1.8
\]
AV block

The graph of the updating function.

\[ u = 1 \]
\[ W = 1 \]
\[ c = 2/3 \]
AV block

Cobweb starting at 1.2 volts.

\[ u = 1 \]
\[ w = 1 \]
\[ c = \frac{2}{3} \]
AV block

Cobweb starting at 1.2 volts.

\[ u=1 \]
\[ W=1 \]
\[ c=2/3 \]
AV block

Cobweb starting at 1.2 volts.

\[ u = 1 \]
\[ W = 1 \]
\[ c = 2/3 \]
AV block

Cobweb starting at 1.2 volts.

\[ u = 1 \]
\[ W = 1 \]
\[ c = 2/3 \]

Graph showing the cobweb with initial conditions and parameters.
AV block

Cobweb starting at 1.2 volts.

\[ u=1 \]
\[ W=1 \]
\[ c=2/3 \]
The potential oscillates between 1.2 and 1.8.
AV block

Between signals the potential decays. Every other signal is ignored and the decay continues until the next signal.
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The Wenckebach phenomenon

We will now study another condition called the Wenckebach phenomenon. Under this condition the heart goes through the following cycle:
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2. The heart skips a beat.
The Wenckebach phenomenon

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3. Normal beating for a while.
4. The heart skips a beat.
The Wenckebach phenomenon

We will now study another condition called the Wenckebach phenomenon. Under this condition the heart goes through the following cycle:

1. Normal beating for a while.
2. The heart skips a beat.
3. Normal beating for a while.
4. The heart skips a beat.
5. And so on.
Let $u = W = 1$. As we have seen, the conditions on an equilibrium $V^*$ are

$$V^* = \frac{u}{1-c} = \frac{1}{1-c} \quad cV^* \leq W$$
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This gives that $cV^* = \frac{c}{1-c}$. Since $W = 1$ we get that $cV^* \leq W$ means that

$$ \frac{c}{1 - c} \leq 1, $$
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that is $c \leq 1 - c$. Hence $2c \leq 1$ and $c \leq 0.5$.
The Wenckebach phenomenon

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that is $c \leq 1 - c$. Hence $2c \leq 1$ and $c \leq 0.5$.

The Wenckebach phenomenon occurs when there is no equilibrium but there almost is one. In other words $0.5 < c$, but $c$ is close to 0.5.
Here is an example where $c = 0.5001$. 

\[
\begin{align*}
\text{u} &= 1 \\
W &= 1 \\
c &= 0.5001
\end{align*}
\]
The Wenckebach phenomenon

Zooming in: there is no equilibrium.

\[ u = 1 \]
\[ W = 1 \]
\[ c = 0.5001 \]
The Wenckebach phenomenon

We cobweb from $V_0 = 1$.

\[ u = 1, \quad W = 1, \quad c = 0.5001 \]
The Wenckebach phenomenon

We cobweb from $V_0 = 1$.

\begin{align*}
u &= 1 \\
W &= 1 \\
c &= 0.5001
\end{align*}
The Wenckebach phenomenon

We cobweb from $V_0 = 1$.

$u=1$

$W=1$

$c=0.5001$
The Wenckebach phenomenon

We cobweb from $V_0 = 1$.

\[ u = 1 \]
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\begin{align*}
&u = 1 \\
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$u = 1$

$W = 1$

$c = 0.5001$
The Wenckebach phenomenon

We cobweb from $V_0 = 1$.

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$u = 1$
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The Wenckebach phenomenon

We cobweb from $V_0 = 1$.

$\begin{align*}
    u &= 1 \\
    W &= 1 \\
    c &= 0.5001
\end{align*}$
The Wenckebach phenomenon

We cobweb from $V_0 = 1$.

\begin{align*}
u &= 1 \\
W &= 1 \\
c &= 0.5001
\end{align*}
The Wenckebach phenomenon

We cobweb from $V_0 = 1$. 

\[ u = 1, \quad W = 1, \quad c = 0.5001 \]
The Wenckebach phenomenon

We cobweb from $V_0 = 1$.

$u=1$
$W=1$
$c=0.5001$
The Wenckebach phenomenon

We cobweb from $V_0 = 1$.

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\[ W = 1 \]
\[ c = 0.5001 \]
We cobweb from $V_0 = 1$.

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\[ W = 1 \]
\[ c = 0.5001 \]
The Wenckebach phenomenon

We cobweb from $V_0 = 1$.

\begin{align*}
u & = 1 \\
W & = 1 \\
c & = 0.5001
\end{align*}
The Wenckebach phenomenon

We cobweb from $V_0 = 1$.

$u=1$
$W=1$
$c=0.5001$
The Wenckebach phenomenon

We cobweb from $V_0 = 1$.

- $u = 1$
- $V = 1$
- $c = 0.5001$
The Wenckebach phenomenon

We cobweb from $V_0 = 1$.

$u=1$
$W=1$
$c=0.5001$
The Wenckebach phenomenon

We cobweb from $V_0 = 1$.

$u = 1$
$W = 1$
$c = 0.5001$
The Wenckebach phenomenon

We cobweb from $V_0 = 1$.

\[ u=1 \]
\[ W=1 \]
\[ c=0.5001 \]
We cobweb from $V_0 = 1$.

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The Wenckebach phenomenon

We cobweb from \( V_0 = 1 \).

\[
\begin{align*}
u &= 1 \\
W &= 1 \\
c &= 0.5001
\end{align*}
\]
The Wenckebach phenomenon

Here is the solution.
The Wenckebach phenomenon

The potential decays between signals from the SA node.