Linear Stability Analysis of a Network of Two Neurons with Delay

Chuan Zhang
Department of Mathematics
Colorado State University
zhang@math.colostate.edu

Report submitted to Prof. I. Oprea for Math 640, Fall 2011

Abstract. This report summarizes the linear stability analysis of the DDE model of a neural network presented in [1]. The network is of Hopfield-type, and consists of two neurons. Following the method described in [1, 2, 8], we implemented linear stability analysis to determine the stability region of the stationary solutions, and simulations confirms the results of the linear stability analysis.

1 Introduction

Interconnected neurons can generate intriguingly rich population dynamics, and some of these population dynamics, such as coherent rhythms, synchronized oscillations, etc., play important roles in various cognitive activities in nervous systems (see for example [3]). In order to understand the underlying mechanisms and functional roles of the population dynamics observed in biological experiments, many different mathematical models have been proposed.

Hopfield-type neural network is a typical class of the mathematical models. In continous network model [5], dynamics of the network is governed by a system of ordinary differential equations:

$$\dot{u}_i(t) = -u_i(t) + \sum_j J_{ij} g_i(u_i) \tag{1.1}$$

The Hopfield neural network was originally proposed to implement content addressable memory [4]. The Cohen-Grossberg-Hopfield convergence theorem [5] guarantees that under the standard assumptions on the sigmoid signal functions g_i and symmetric connection J_{ij}, every solution of the system (1.1) converges to the set of equilibria. In order to include more complicated dynamics into the network model, biologically more plausible assumptions, such as asymmetric connections or synaptic transmission delays etc., need to be imposed into the network model.

This report considers the delay differential equation model of a Hopfield-type neural network of two neurons. A detailed linear stability analysis of this network model has been presented in [1]. In this report, we summarize the stability analysis presented in [1], and verify the analytic results with numerical simulations.

2 Preliminaries

2.1 Notation

Follow [1] and [2], in this report, we adopt the following notations. Let $\mathbb{R}^n$ denote n dimensional Euclidean space with norm $|\cdot|$. For $b > a$, denote $C([a,b],\mathbb{R}^n)$, the Banach space of continuous functions from $[a,b]$ to $\mathbb{R}^n$ with the topology of uniform convergence, i.e., for $\phi \in C([a,b],\mathbb{R}^n)$, the norm of ϕ is defined as

$$\|\phi\| = \sup_{a\leq\theta\leq b} |\phi(\theta)|.$$

When $\tau > 0$ is fixed, $C([-\tau,0],\mathbb{R}^n)$ is denoted C. Let $\Omega \subseteq \mathbb{R}\times C$, $f : \Omega \to \mathbb{R}^n$ be a given function, and "$\cdot$" be the *right-hand derivative*, then

$$\dot{x}(t) = f(t, x_t) \tag{2.1}$$

is called a *retarded delay differential equation* (RDDE) or a *retarded functional differential equation* (RFDE) on Ω, where $x_t \in C$ is defined as $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$. Some authors also call the equation (2.1) a *delay differential equation*.

A function $x(t)$ is called a *solution* of Eq. (2.1) on $[\sigma-\tau, \sigma+A)$ if $x \in C([\sigma-\tau,\sigma+A),\mathbb{R}^n)$, $(t, x_t) \in \Omega$ and x_t satisfies Eq. (2.1) for $t \in [\sigma, \sigma+A)$. For given $\sigma \in \mathbb{R}$, $\phi \in C$, $x(\sigma,\phi)$ is said to be a *solution of (2.1) with initial value ϕ at σ*, or simply a solution through (σ,ϕ), if there is an $A > 0$ such that $x(\sigma,\phi)$ is a solution of (2.1) on $[\sigma-\tau,\sigma+A)$ and $x_\sigma(\sigma,\phi) = \phi$.

Example 1. Consider the scalar equation

$$\dot{x}(t) = -2x(t) + x(t-\pi/2) + \sin(t), \text{ for } t \geq 0 \tag{2.2}$$

with

$$x(t) = \phi(t), \text{ for } -\pi/2 \leq t \leq 0 \tag{2.3}$$

where ϕ is a given continuous function.

With the above notation, we have $f(t, x_t) = -2x_t(0)+x_t(-\pi/2)+\sin(t)$. Following the idea used for linear ordinary differential equations with constant coefficients, we seek a particular solution of Eq. (2.2) of the form

$$\tilde{x}(t) = c_1\cos(t) + c_2\sin(t).$$

Substituting it to (2.2) gives

$$\begin{aligned} -c_1\sin(t) + c_2\cos(t) &= -2c_1\cos(t) - 2c_2\sin(t) + c_1\cos(t-\pi/2) + c_2\sin(t-\pi/2) + \sin(t) \\ &= -2c_1\cos(t) - 2c_2\sin(t) + c_1\sin(t) - c_2\cos(t) + \sin(t) \\ &= -(2c_1 + c_2)\cos(t) - (2c_2 - c_1 - 1)\sin(t) \end{aligned}$$

it follows that

$$\begin{cases} 2c_2 - 2c_1 = 1 \\ 2c_2 + 2c_1 = 0 \end{cases}$$

i.e., $c_1 = -1/4$, $c_2 = 1/4$. So a solution of (2.2) is defined by

$$\tilde{x}(t) = -1/4\cos(t) + 1/4\sin(t).$$

Let x be the solution of (2.2) and (2.3) on $[-\pi/2, \infty)$, and define $y = x - \tilde{x}$. Then y satisfies the homogeneous equation

$$\dot{y}(t) = -2y(t) + y(t - \pi/2), \text{ for } t \geq 0$$

with

$$y(t) = \phi(t) + 1/4\cos(t) - 1/4\sin(t), \text{ for } -\pi/2 \leq t \leq 0.$$

Thus, x is given by

$$x(t) = \tilde{x}(t) + y(t), \text{ for } t \geq -\pi/2.$$

Therefore, with the above notation, we have $x(0, \phi) = \tilde{x}(t) + y(t)$.

If the righthand side of (2.1) is independent of t then the system is said to be *autonomous*. In this case we consider σ in the notations to be equal to zero.

2.2 Existence, Uniqueness, and Continuous Dependence of Solutions

For the completeness of the presentation, we present the four fundamental theorems. Proofs can be found in textbooks such as [6], and [7] etc.

Theorem 1. *(**Existence**) In (2.1) suppose Ω is an open subset of $\mathbb{R} \times C$ and f is continuous on Ω. If $(\sigma, \phi) \in \Omega$ then there is a solution of (2.1) passing through (σ, ϕ).*

Theorem 2. *(**Uniqueness**) Suppose Ω is open in $\mathbb{R} \times C$, $f : \Omega \to \mathbb{R}^n$ is continuous and $f(t, \phi)$ is Lipschitz in ϕ for each compact subset in Ω. If $(\sigma, \phi) \in \Omega$ then there is a unique solution of (2.1) passing through (σ, ϕ).*

Theorem 3. *(**Continuous Dependence**) Suppose $\Omega \subset \mathbb{R} \times C$ is open, $(\sigma, \phi) \in \Omega$, $f \in \mathcal{C}(\Omega, \mathbb{R}^n)$, and x is a solution of Eq. (2.1) through (σ, ϕ), which exists and is unique on $[\sigma - \tau, b]$, $b > \sigma - \tau$. Let $W \subseteq \Omega$ be the compact set defined by*

$$W = \{(t, x_t) : t \in [\sigma, b]\},$$

and let V be a neighborhood of W on which f is bounded. If (σ^k, ϕ^k, f^k), $k \in \mathbb{N}$, satisfies $\sigma^k \to \sigma$, $\phi^k \to \phi$, and $|f^k - f|_V \to 0$ as $k \to \infty$, then there is a K such that, for $k \geq K$, each solution $x^k = x^k(\sigma^k, \phi^k, f^k)$ through (σ^k, ϕ^k) of

$$\dot{x}(t) = f^k(t, x_t)$$

exists on $[\sigma^k - \tau, b]$ and $x^k \to x$ uniformly on $[\sigma - \tau, b]$.

Remark 1. Since some x^k may not be defined on $[\sigma-\tau, b]$, the statement "$x^k \to x$ uniformly on $[\sigma, \phi]$" means that for any $\epsilon > 0$, there is a $k_1(\epsilon)$ such that $x^k(t)$ is defined on $[\sigma-\tau+\epsilon, b]$ and $x^k \to x$ uniformly on $[\sigma-\tau+\epsilon, b]$ for all $k \geq k_1(\epsilon)$.

Let x be a solution of Eq. (2.1) on an interval $[\sigma, a)$, $a > \sigma$. We say $\hat{x}$ is a *continuation* of x if there is a $b > a$ such that $\hat{x}$ is defined on $[\sigma-\tau, b)$, coincides with x on $[\sigma-\tau, a)$, and x satisfies Eq. (2.1) on $[\sigma, b)$. A solution x is *noncontinuable* if no such continuation exists; that is, the interval $[\sigma, a)$ is the *maximal interval of existence* of the solution x.

Theorem 4. *Suppose Ω is an open set in $\mathbb{R} \times C$, $f : \Omega \to \mathbb{R}^n$ is completely continuous*[1]*, and x is a noncontinuable solution of Eq. (2.1) on $[\sigma-\tau, b)$. Then for any closed bounded set U in $\mathbb{R} \times C$, $U \subset \Omega$, there is a t_U such that $(t, x_t) \notin U$ for $t_U \leq t < b$.*

Remark 2. Theorem 4 says that solution of Eq. (2.1) either exists for all $t \geq \sigma$ or becomes unbounded (with respect to Ω) in finite time.

2.3 The System

The system studied in [1] is

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + \alpha_{11}\tanh(x_1(t-\tau_1)) + \alpha_{12}\tanh(x_2(t-\tau_2)) \\ \dot{x}_2(t) = -x_2(t) + \alpha_{21}\tanh(x_1(t-\tau_1)) + \alpha_{22}\tanh(x_2(t-\tau_2)) \end{cases}$$

In this report, we only consider a special case of the above system in which $\tau_1 = \tau_2 = \tau$, and demonstrate the basic ideas of the linear stability analysis. Therefore, the system under considerations in this report is

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + \alpha_{11}\tanh(x_1(t-\tau)) + \alpha_{12}\tanh(x_2(t-\tau)) \\ \dot{x}_2(t) = -x_2(t) + \alpha_{21}\tanh(x_1(t-\tau)) + \alpha_{22}\tanh(x_2(t-\tau)) \end{cases} \tag{2.4}$$

Clearly, the above system (2.4) is autonomous, and the vector field f is defined as

$$f(\phi) = \begin{pmatrix} f_1(\phi_1, \phi_2) \\ f_2(\phi_1, \phi_2) \end{pmatrix} = \begin{pmatrix} -\phi_1(0) + \alpha_{11}\tanh(\phi_1(-\tau)) + \alpha_{12}\tanh(\phi_2(-\tau)) \\ -\phi_2(0) + \alpha_{21}\tanh(\phi_1(-\tau)) + \alpha_{22}\tanh(\phi_2(-\tau)) \end{pmatrix}$$

In this report, $f \in C([-\tau, 0], \mathbb{R}^2)$ is infinitely differentiable on the entire function space, therefore, all of the first three theorems in the preceding section apply. It follows from theorem 4 that solutions of (2.4) exist for all positive time. A direct verification shows that the system is symmetric about the origin, i.e., it is invariant under the symmetry $(x_1, x_2) \to (-x_1, -x_2)$, and changing the sign of α_{12} and α_{21} does not change the qualitative behavior of the system.

[1] A function f is said to be *completely continuous*, if f is continuous and takes closed bounded sets into compact sets.

2.4 Stability and Stability Criteria

2.4.1 Definition of Stabilities

In this section, we follow [2, 8] to formally define stability of a solution, and introduce the stability criteria for DDEs that we will use to analyze our system.

Definition 1. Suppose the system of delay differential equations (2.1) has an equilibrium point at the origin, i.e., $f(t, 0) = 0$ for all $t \in \mathbb{R}$, and $f : \mathbb{R} \times C \to \mathbb{R}^n$ is completely continuous. Then, the trivial solution $x = 0$ of (2.1) is said to be *stable* in the Lyapunov sense, if for every $\epsilon > 0$ and $\sigma \in \mathbb{R}$, there exists a $\delta = \delta(\epsilon, \sigma)$ such that $\|x_t(\sigma, \phi)\| \leq \epsilon$ for all $t \geq \sigma$ and for any initial function ϕ satisfying $\|\phi\| < \delta$. Moreover, if the trivial solution is stable and δ does not depend on σ, it is said to be *uniformly stable*. The trivial solution is called *asymptotically stable* if it is stable and for every $\sigma \in \mathbb{R}$, there exists a $\Delta = \Delta(\sigma)$ such that $\lim_{t\to\infty} \|x_t(\sigma, \phi)\| = 0$ for any ϕ satisfying $\|\phi\| < \Delta$. Furthermore, if the trivial solution is asymptotically stable, and Δ does not depend on σ, it is called *uniformly asymptotically stable*.

Remark 3. When periodic solutions of autonomous delay differential equations system is considered, the *orbital stability* will be used; and if the system contains parameters, say delays, the *structural stability* will be used.

2.4.2 Characteristic Functions and Stability

Similar to the local stability theory for ODEs, most studies on DDEs also start from the local stability analysis of some special solutions, and *constant solution* (also known as *stationary solution*) is usually investigated first. For the autonomous DDEs, the linearized equations in a neighborhood of the constant solution depends on the locations of the roots of the associated characteristic equation. Here we follow [1, 2, 8] to formally define the characteristic function as follows.

Suppose the linearized autonomous and homogeneous DDEs about the trivial solution $x = 0$ is

$$\dot{x}(t) = L(x_t) \tag{2.5}$$

where the functional $L : C \to \mathbb{R}^n$ is continuous and linear. Then the function given by

$$D(\lambda) = \det(\lambda I - L(e^{\lambda t} I)),\ \lambda \in \mathbb{C} \tag{2.6}$$

is called the *characteristic function* corresponding to the linear autonomous system (2.5), where I is the $n \times n$ unit matrix, and solutions λ to the *characteristic equation*, $D(\lambda) = 0$ are referred to as the *eigenvalues* of the system.

Definition 2. The characteristic function D of a system of DDEs is called *stable* if and only if it contains no eigenvalue with non-negative real part.

Theorem 5. *If* $\sup\{Re\lambda | D(\lambda) = 0\} < 0$, *then the trivial solution of (2.5) is uniformly asymptotically stable. If* $Re\lambda \geq 0$ *for some* λ *satisfying the characteristic equation, then (2.5) is unstable.*

Theorem 6. *If the trivial solution of (2.5) is uniformly asymptotically stable, then the trivial solution of (2.1) is also uniformly asymptotically stable. If $Re\lambda > 0$ for some λ satisfying the characteristic equation then the trivial solution of (2.1) is unstable.*

While several stability criteria have been used (see section 1.3 in [8]), in this report, we only adopt the stability criterion based directly on the investigation of the characteristic function in its original form [1, 2, 8], which was described in the above theorems 5 and 6.

Remark 4. The stability of the trivial solution of the system (2.5) depends on the locations of the roots of the associated characteristic equation. When delays are finite, the characteristic equations are functions of delays, and so be the roots of these characteristic equations. As long as delays change, the stability of the trivial solution may change accordingly. This phenomenon is called *stability switch.*

In next section, we analyze the stability switch of the system of DDEs (2.4).

3 Linear Stability Analysis and Bifurcations

3.1 Characteristic Equation of the Linearized System

Consider the system (2.4). Suppose $x^* = (x_1^*, x_2^*)$ is a stationary solution of the system. Let the system be disturbed from the equilibrium by a small perturbation which lasts from $t = \sigma - \tau$ to σ. Suppose $\delta x(t) = (\delta x_1(t), \delta x_2(t))$ is the deviation from the equilibrium, i.e. $x(t) = x^* + \delta x(t)$. Thus, we get

$$\begin{cases} \dot{x}_1(t) = \dot{\delta x}_1(t) = -x_1^* - \delta x_1(t) + \alpha_{11}\tanh(x_1^* + \delta x_1(t-\tau)) + \alpha_{12}\tanh(x_2^* + \delta x_2(t-\tau)) \\ \dot{x}_2(t) = \dot{\delta x}_2(t) = -x_2^* - \delta x_2(t) + \alpha_{21}\tanh(x_1^* + \delta x_1(t-\tau)) + \alpha_{22}\tanh(x_2^* + \delta x_2(t-\tau)) \end{cases}$$

Direct substitution shows that $x^* = (x_1^*, x_2^*) = 0$ is a stationary solution of the system (2.4), in this report, we only consider the trivial stationary solution. For the nontrivial stationary solutions, we may translate them to the origin by appropriate change of coordinates. Thus, applying the Taylor expansion of the system (2.4) around the trivial stationary solution $(0, 0)$ yields the following linearized system

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + \alpha_{11}x_1(t-\tau) + \alpha_{12}x_2(t-\tau) \\ \dot{x}_2(t) = -x_2(t) + \alpha_{21}x_1(t-\tau) + \alpha_{22}x_2(t-\tau) \end{cases} \tag{3.1}$$

To study the stability of the above system, we seek the solution of the form $x(t) = e^{\lambda t}(c_1, c_2)^T$, where $c_1, c_2 \in \mathbb{R}$ are constants, and $\lambda \in \mathbb{C}$. Direct substitution of $x(t)$ into (3.1) yields

$$\begin{cases} c_1\lambda e^{\lambda t} = -c_1 e^{\lambda t} + \alpha_{11}c_1 e^{\lambda(t-\tau)} + \alpha_{12}c_2 e^{\lambda(t-\tau)} \\ c_2\lambda e^{\lambda t} = -c_2 e^{\lambda t} + \alpha_{21}c_1 e^{\lambda(t-\tau)} + \alpha_{22}c_2 e^{\lambda(t-\tau)} \end{cases}$$

i.e.,

$$\begin{pmatrix} (\lambda+1) - \alpha_{11}e^{-\lambda\tau} & \alpha_{12}e^{-\lambda\tau} \\ \alpha_{21}e^{-\lambda\tau} & (\lambda+1) - \alpha_{22}e^{-\lambda\tau} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.2}$$

The above system (3.2) has nontrivial solutions for (c_1, c_2) if and only if

$$\det\begin{pmatrix} (\lambda+1)-\alpha_{11}e^{-\lambda\tau} & \alpha_{12}e^{-\lambda\tau} \\ \alpha_{21}e^{-\lambda\tau} & (\lambda+1)-\alpha_{22}e^{-\lambda\tau} \end{pmatrix} = 0$$

i.e.

$$((\lambda+1)e^{\lambda\tau})^2 - (\alpha_{11}+\alpha_{22})e^{\lambda\tau} + (\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}) = 0 \tag{3.3}$$

Solving the above characteristic equation (3.3) yields

$$(\lambda+1)e^{\lambda\tau} = \xi \pm \sqrt{\xi^2 - \eta} \tag{3.4}$$

where $\xi = \dfrac{\alpha_{11}+\alpha_{22}}{2}$ and $\eta = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$.

Let $\lambda = \rho + i\omega$, substituting it into (3.4) gives

$$(\rho + 1 + i\omega)e^{\rho\tau}(\cos(\omega\tau) + i\sin(\omega\tau)) = \xi \pm \sqrt{\xi^2 - \eta} \tag{3.5}$$

Since ξ and η are half of the trace of the connection matrix and determinant of the connection matrix respectively, they are determined by the interneuron connections of the network. in next two subsections, we study the stability of the trivial solution in two different cases, in which the connection matrix has real eigenvalues and complex eigenvalues respectively.

3.2 Connection Matrices with Real Eigenvalues

Suppose the connection matrix of the network has real eigenvalues, i.e., $\xi^2 \geq \eta$, then

$$\begin{cases} ((\rho+1)\cos(\omega\tau) - \omega\sin(\omega\tau))e^{\rho\tau} = \xi \pm \sqrt{\xi^2-\eta} \\ (\rho+1)\sin(\omega\tau) + \omega\cos(\omega\tau) = 0 \end{cases} \tag{3.6}$$

Thus, from the second equation of (3.6), it follows that $\rho = -(1 + \omega\cot(\omega\tau))$. Since the system changes its stability at $\rho = 0$, the solutions of the equation $1 + \omega\cot(\omega\tau) = 0$ give the positions where the system loses/attains stability. Therefore, $\tilde{\omega}/\tau = -\tan(\tilde{\omega})$, where $\tilde{\omega} = \omega\tau$. As the equation is transcendental, the solutions only can be obtained numerically. Figure 1 below illustrates the lines $y = \tilde{\omega}/\tau$ and the curve $y = -\tan(\tilde{\omega})$, and the intersections of these lines with the curve correspond to the solutions.

For $\rho = 0$, to understand how the system changes stability, we discuss two different cases, $\omega = 0$ and $\omega \neq 0$. If $\omega = 0$, then (3.6) becomes

$$\xi \pm \sqrt{\xi^2 - \eta} = 1$$

which implies that

$$\eta = 2\xi - 1 \tag{3.7}$$

[1] pointed out that this is the line at which the non-trivial stationary solutions come into existence, and concluded that crossing this line results in a pitchfork bifurcation. To determine the stability of

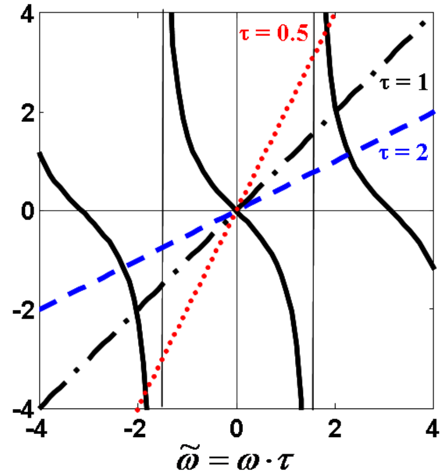

Figure 1: Solutions of $\tilde{\omega}/\tau = -\tan(\tilde{\omega})$, which geometrically correspond to the intersections of the lines $y = \tilde{\omega}/\tau$ and the curve $y = -\tan\omega$. In the above figure, the red dot line corresponds to $\tau = 0.5$, the black dot-dashed line corresponds to $\tau = 1$, and the blue dashed line corresponds to $\tau = 2$.

the non-trivial solutions created at the pitchfork bifurcation, the partial derivative of the eigenvalue of the system, $\lambda = \rho + i\omega$, with respect to η was calculated. According to (3.6), we have that

$$\begin{aligned}\tau\tfrac{\partial\rho}{\partial\eta}e^{\rho\tau}[(\rho+1)\cos(\omega\tau) - \omega\sin(\omega\tau)] + e^{\rho\tau}[\tfrac{\partial\rho}{\partial\eta}\cos(\omega\tau) - (\rho+1)\tau\tfrac{\partial\omega}{\partial\eta}\sin(\omega\tau)\\ -\tfrac{\partial\omega}{\partial\eta}\sin(\omega\tau) - \omega\tau\tfrac{\partial\omega}{\partial\eta}\cos(\omega\tau)] = \tfrac{-1}{2\sqrt{\xi^2-\eta}}\end{aligned}$$

and

$$\begin{aligned}\tau\tfrac{\partial\rho}{\partial\eta}e^{\rho\tau}[\omega\cos(\omega\tau) + (\rho+1)\sin(\omega\tau)] + e^{\rho\tau}[\tfrac{\partial\omega}{\partial\eta}\cos(\omega\tau) - \omega\tau\tfrac{\partial\omega}{\partial\eta}\sin(\omega\tau)\\ +\tfrac{\partial\rho}{\partial\eta}\sin(\omega\tau) + (\rho+1)\tau\tfrac{\partial\omega}{\partial\eta}\cos(\omega\tau)] = 0\end{aligned}$$

At $\lambda = \rho + i\omega = 0$, $\rho = \omega = 0$, then

$$(\tau+1)\frac{\partial\rho}{\partial\eta} = \frac{-1}{2\sqrt{\xi^2-\eta}}$$

Since $\tau + 1 \geq 1$ and $\sqrt{\xi^2-\eta} > 0$, it follows that $\dfrac{\partial\rho}{\partial\eta} < 0$, which implies that when we move across the boundary at the pitchfork bifurcation the origin loses stability, for η is decreasing as we move across the boundary. Moreover, [1] showed that the non-trivial stationary solutions of the system created at the pitchfork bifurcation remain bounded, it follows that the non-trivial stationary solutions are stable.

When $\omega \neq 0$, replacing the term $-\omega$ in the first equation of (3.6) by $\tan(\omega\tau)$ yields

$$\eta = 2\xi\sec(\omega\tau) - \sec^2(\omega\tau) \tag{3.8}$$

By using the center manifold reduction, [1] showed that crossing the above line when $\xi > \sec(\omega\tau)$ results in a Hopf bifurcation.

At the points of intersection between the lines $\eta = 2\xi - 1$ and $\eta = 2\xi\sec(\omega\tau) - \sec^2(\omega\tau)$, [1] showed that the Hopf bifurcation interacts with pitchfork bifurcations.

3.3 Connection Matrices with Complex Eigenvalues

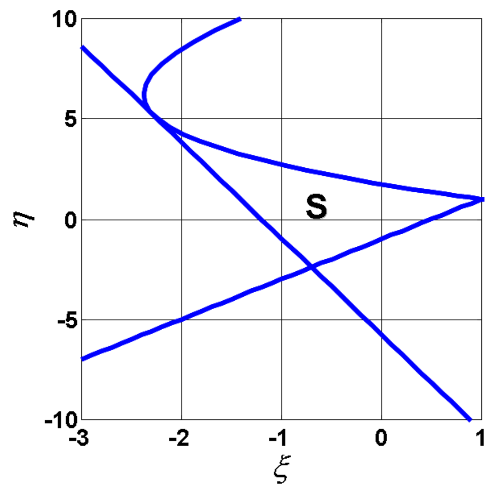

Figure 2: Stability region of the trivial solution ($\tau = 1$). The trivial stationary solution of the system with parameters (ξ, η) taken from the region S is stable, and that of the system with parameters taken from the other regions is unstable.

Suppose the connection matrix of the network has complex eigenvalues, i.e., $\xi^2 < \eta$, then

$$\begin{cases} [(\rho+1)\cos(\omega\tau) - \omega\sin(\omega\tau)]e^{\rho\tau} = \xi \\ [(\rho+1)\sin(\omega\tau) + \omega\cos(\omega\tau)]e^{\rho\tau} = \pi\sqrt{\eta - \xi^2} \end{cases} \tag{3.9}$$

Let $\rho = 0$, then

$$\begin{cases} \cos(\omega\tau) - \omega\sin(\omega\tau) = T \\ \omega\cos(\omega\tau) + \sin(\omega\tau) = \pm\sqrt{\eta - \xi^2} \end{cases} \tag{3.10}$$

Squaring both of the above two equations and add them together gives $1 + \omega^2 = \eta$, i.e.

$$\omega = \sqrt{\eta - 1} \tag{3.11}$$

Substituting it back to the first equation in (3.10) yields

$$\xi = \cos(\tau\sqrt{\eta - 1}) - \sqrt{\eta - 1}\sin(\tau\sqrt{\eta - 1})$$

Figure 2 shows the stability regions bounded by the four curves obtained in the preceding two subsections.

4 Simulations

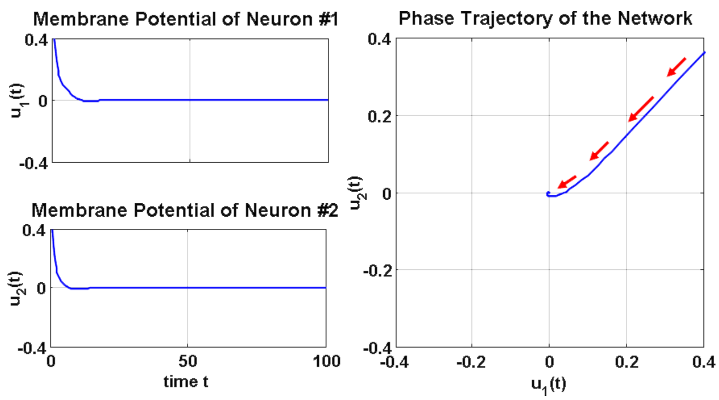

Figure 3: Simulation of the Hopfield-type neural network of two neurons with delay $\tau = 1$. Details on parameters setting see text.

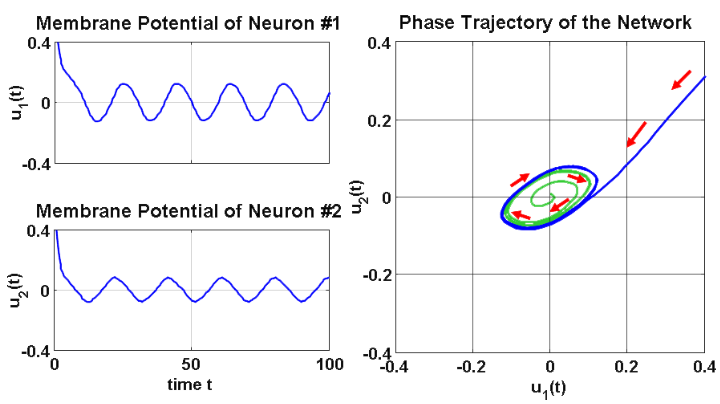

Figure 4: Simulation of the Hopfield-type neural network of two neurons with delay $\tau = 1$. Details on parameters setting see text.

In this section we implement the numerical simulations to verify the above linear stability analysis. In stead of the system (2.4), we consider a slightly changed system, which is more relevant to my ongoing research project [9]. The system is given as follows.

$$\begin{cases} \dot{u}_1(t) = -u_1(t) + C_0\beta_K J_{11}\tanh(\lambda u_1(t-\tau)) + C_1\beta_K J_{12}\tanh(\lambda u_2(t-\tau)) \\ \dot{u}_2(t) = -u_2(t) + C_0\beta_K J_{21}\tanh(\lambda u_1(t-\tau)) + C_1\beta_K J_{22}\tanh(\lambda u_2(t-\tau)) \end{cases} \tag{4.1}$$

where $C_0 + C_1 = 1$, $\beta_K = \beta/\lambda$, and $\beta \geq 1$. In the following simulations, we fix the parameters $C_0 = 0.3$, $\lambda = 10$, $\tau = 1$, the initial condition $\phi_1(t) = 1$, $\phi_2(t) = 1$ for $-1 \leq t \leq 0$, and vary

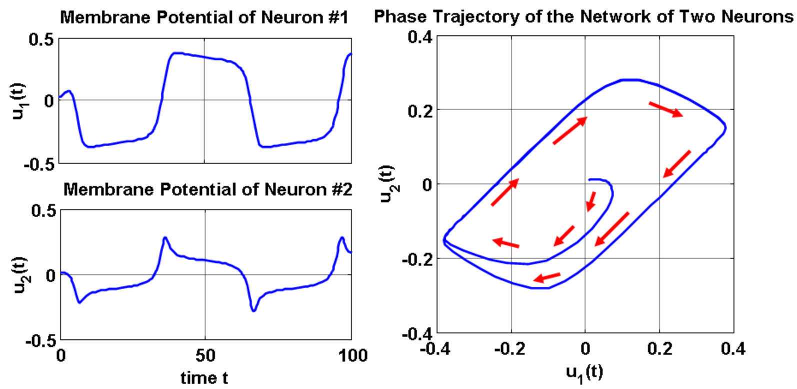

Figure 5: Simulation of the Hopfield-type neural network of two neurons with delay $\tau = 1$. Details on parameters setting see text.

the parameter β and the connection matrix J to study the changes in the stability of the trivial stationary solution $(0, 0)$.

Following the linearization introduced in the preceding section, we obtain the linearized system:

$$\begin{cases} \dot{u}_1(t) = -u_1(t) + C_0\beta J_{11}u_1(t-\tau) + C_1\beta J_{12}u_2(t-\tau) \\ \dot{u}_2(t) = -u_2(t) + C_0\beta J_{21}u_1(t-\tau) + C_1\beta J_{22}u_2(t-\tau) \end{cases} \tag{4.2}$$

From the linearized system (4.2), we can see that changing the parameter β and the connection matrix J is indeed equivalent to the changes in α_{11}, α_{12}, α_{21} and α_{22} in the original system (2.4) considered in [1].

First, we take $\beta = 1, 2, 4$, and numerically solve the corresponding systems respectively. Figures 3, 4, and 5, show the corresponding solutions $u_1(t)$, $u_2(t)$ and phase trajectories.

5 Summary

In this report, we summarize the linear stability analysis of the DDE model of a neural network presented in [1]. While several stability criteria have been proposed [8], we followed [1] to use the criterion based directly on the investigation of the characteristic function to study the stability of the stationary solutions of the system. The basic idea is that first linearize the system about the origin (assume the origin is a stationary solution of the system, if it is not, we could move the stationary solution to the origin through a linear coordinate transformation), then substitute the trial solution $x(t) = e^{\lambda t}c$, where $c \in \mathbb{R}^n$ is a constant n-dimensional vector, into the linearized system to get the characteristic equation. Unfortunately however, usually, the characteristic equation is a transcendental equation, which is impossible to be solved analytically. Some authors used numerical methods to solve the problems. In this report, we followed [1], instead of solving for the characteristic roots directly, we assume the characteristic roots have the form $\lambda = \rho + i\omega$, and apply the stability criteria on the characteristic roots to partition the parameter space into stable/unstable regions. On the boundary of every region, the stability of the corresponding solutions changes. [1] showed that for the network of two neurons, the trivial stationary solution could lose stability through either a pitchfork bifurcation or a Hopf bifurcation. Also following from [1], we implement some numerical simulations, and the results confirm the results of the linear stability analysis.

Appendix: Matlab Codes for Simulations Presented in Section 4

```
function sol = N2D1_01
clear all;
close all;
clc;

global C0 C1 Beta Lambda J0 J1;
C0 = 0.3; C1 = 1 - C0;
Beta = 2; Lambda = 10;
J0 = [ 1  0;  0  1];
J1 = [ 0  1; -1  0];
J  = J0 + J1;

tau1 = 1; tau2 = tau1;
t0 = 0; t1 = 100;
umin = -0.4; umax = 0.4;

sol = dde23('N2D1f',[tau1, tau2],ones(2,1).*1,[t0, t1],[],C0,C1,Beta,Lambda,J);
fig1 = figure(1);
subplot(2,2,1);
    plot(sol.x,sol.y(1,:),'-b','LineWidth',2);
    title('Membrane Potential of Neuron #1')
    xlabel('time t');
    ylabel('u_1(t)');
    axis([t0 t1 umin umax]);
    grid on;
subplot(2,2,3);
    plot(sol.x,sol.y(2,:),'-b','LineWidth',2);
    title('Membrane Potential of Neuron #2')
    xlabel('time t');
    ylabel('u_2(t)');
    axis([t0 t1 umin umax]);
    grid on;
subplot(2,2,[2 4]);
    plot(sol.y(1,:),sol.y(2,:),'-b','LineWidth',2);
    title('Phase Trajectory of the Network')
    xlabel('u_1(t)');
    ylabel('u_2(t)');
    axis([umin umax umin umax]);
    axis square;
    grid on;
```

```
function uh = N2D1Hist(t,lambda)
%N2D1Hist  The history function for the Hopfield-type Neural Network of Two
%          Neurons with one constant delay.
uh = ones(2,1).*0.1;


function yp = N2D1f(t,y,Z,C0,C1,Beta,Lambda,J)
%N2D1f  The derivative function for the Hopfield-type Neural Network of Two
%       Neurons with one constant delay.
ylag1 = Z(1,1);
ylag2 = Z(2,1);
BetaK = Beta/Lambda;
yp = [
        -y(1) + C0*BetaK*J(1,1)*tanh(Lambda*ylag1) + C1*BetaK*J(1,2)*tanh(Lambda*ylag2);
        -y(2) + C0*BetaK*J(2,1)*tanh(Lambda*ylag1) + C1*BetaK*J(2,2)*tanh(Lambda*ylag2);
     ];
```

References

[1] Olien, L. [1995] *Analysis of a Delay Differential Equation Model of a Neural Network*. Thesis, Department of Mathematics and Statistics, McGill University, Montreal. (*Chapters 2 and 3*)

[2] Kuang, Y. [1993] *Delay Differential Equations with Applications in Population Dynamics*. Academic Press: San Diego. (*sections 2.1∼2.4, 2.8∼2.9, and 3.1∼3.2*)

[3] Gray C.M. [1994] *Synchronous Oscillations in Neuronal Systems: Mechanisms and Functions*. Journal of Computational Neuroscience, vol. 1: 11-38.

[4] Hopfield, J.J. [1982] *Neural networks and physical systems with emergent collective computational abilities*. Proc. Natl. Acad. Sci. USA vol.79:2554-2558.

[5] Hopfield, J.J. [1984] *Neural networks and physical systems with emergent collective computational abilities*. Proc. Natl. Acad. Sci. USA vol.79:2554-2558.

[6] Driver, R.D. [1977] *Ordinary and Delay Differential Equations*. Springer-Verlag: New York, Heidelberg, Berlin. (*sections 21∼34*)

[7] Hale, J. & Verduyn Lunel, S. [1993] *Introduction to Functional Differential Equations*. Springer-Verlag: New York, Heidelberg, Berlin.

[8] Stépán, G. [1989] *Retarded Dynamical Systems: Stability and Characteristic Functions*. Pitman Research Notes in Math. Series **210**, Longman Scientific & Technical.(*sections 1.1∼1.3*)

[9] Zhang, C., Dangelmayr, G., & Oprea, I. [2011] *Storing Cycles in Continuous Asymmetric Hopfield-type Networks*. In preparation.