Dynamics at the Horsetooth Volume 2, 2010.

A Duffing Family of Dynamical Systems

David Hopkins and Mike Mikucki
Department of Mathematics
Colorado State University hopkins@math.colostate.edu,mikucki@math.colostate.edu

Report submitted to Prof. P. Shipman for Math 540, Fall 2010

Abstract. In this paper, we introduce a set of equations that we will call the Duffing Family. Each member of the Duffing Family is a polynomial extension of the usual Duffing equation. We investigate the properties of chaos, bifurcations, and shift map conjugacy in the extended system. We claim without proof that each member of the Duffing Family exhibits chaos under certain parameters and is topologically conjugate to the shift map on n indices.

Keywords: Duffing, chaos, shift map

1 Introduction

In this paper, we introduce a set of equations that we will call the Duffing Family. These equations are extensions of the Duffing equation,

$$\ddot{x} + b\dot{x} - x + x^3 = \Gamma \cos(\omega t). \tag{1}$$

We aim to generalize the Duffing equation so that chaos exists in the vicinity of more than three fixed points.

2 Generalizations and Restrictions

We first generalize the form of the Duffing equation to

$$\ddot{x} + b\dot{x} + p(x, \dot{x}) = \Gamma \cos(\omega t), \tag{2}$$

where $p(x, \dot{x})$ is a polynomial function. By setting $x = x$ and $y = \dot{x}$, the two dimensional version of the Duffing equation is

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -by + \Gamma \cos(\omega t) - p(x, y). \end{aligned}$$

For the original Duffing equation with three fixed points, $p(x, y) = x^3 - x$. For the rest of this paper, we will refer to the original Duffing equation as D_2, denoting the number of spirals or centers the system contains. When $\Gamma = 0$, three fixed points occur at $x = -1$, $x = 0$, and $x = 1$. The origin is a saddle point and $x = \pm 1$ are spirals or centers, depending on the value of b. Any trajectory will then be pulled toward the origin by the stable manifold and by possibly by the spiral dynamics, but pushed away by the unstable manifold of the origin and possibly the spiral dynamics. A plot with $b = -0.1$, producing unstable spirals is shown in Figure 1.

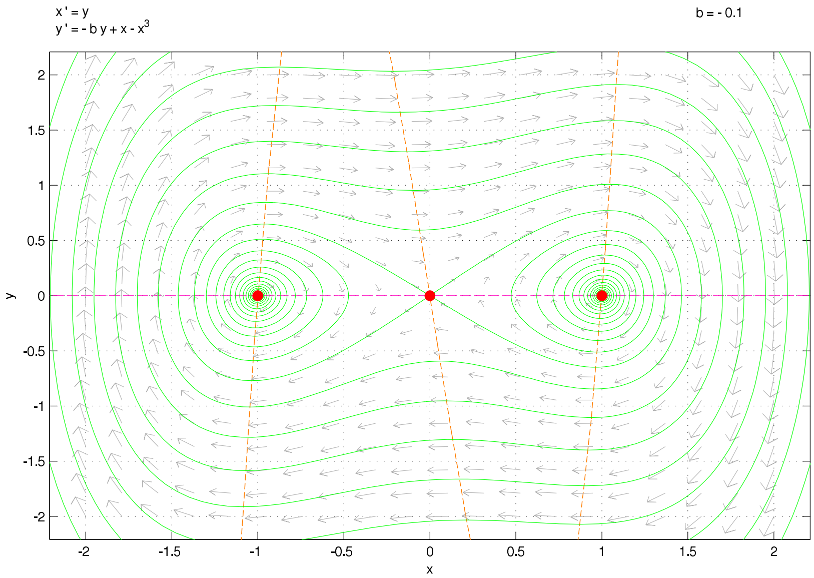

Figure 1: D_2 in the xy-plane with $b = -0.1$. Note that the orange line is the y-nullcline and the green lines are sample orbits.

To ensure that two spirals or centers and one saddle point govern the dynamics, certain restrictions must be satisfied by the parameter b. We linearize about each fixed point to determine these restrictions.

The Jacobian of the general Duffing equation is

$$\mathrm{J}f(x,y) = \begin{pmatrix} 0 & 1 \\ -\frac{\partial p}{\partial x} & -b - \frac{\partial p}{\partial y} \end{pmatrix}. \tag{3}$$

Therefore, to ensure that $x = \pm 1$ are spirals or centers and that $x = 0$ is a saddle, the following restrictions must be satisfied:

Restriction	Reason
1-2. $\frac{\partial p}{\partial x}\|_{x=\pm 1} > 0$	$x = \pm 1$ are not saddle points
3. $\frac{\partial p}{\partial x}\|_{x=0} < 0$	$x = 0$ is a saddle point
4-5. $\left((-b - \frac{\partial p}{\partial y})^2 - 4\frac{\partial p}{\partial x}\right)\|_{x=\pm 1} \geq 0$	$x = \pm 1$ are spirals or centers
6-8. $p(x, \dot{x})\|_{x=0,x=\pm 1} = 0$	$x = 0$ and $x = \pm 1$ are fixed points

For D_2, $p(x, y) = x^3 - x$ with $|b| \leq \sqrt{8}$ satisfies all eight restrictions. We will investigate functions $p(x, y)$ that satisfy these types of restrictions in a higher-order Duffing equation.

3 Duffing Equation with Three Spirals

We seek a Duffing equation with five fixed points, which we will call D_3. In order for (x, y) to be a fixed point, we require $y = 0$, since $\dot{x} = y$. Therefore, all fixed points must lie on the x-axis. A natural extension is to add fixed points $x = \pm 2$ to the other three. We require the fixed points $x = 0$ and $x = \pm 2$ to be spirals or centers, and the fixed points $x = \pm 1$ to be saddle points. Using the same Jacobian, we find the following restrictions on $p(x, y)$.

Restriction	Reason
1-3. $\frac{\partial p}{\partial x}\|_{x=\pm 2,x=0} > 0$	$x = \pm 2$ and $x = 0$ are not saddle points
4-5. $\frac{\partial p}{\partial x}\|_{x=\pm 1} < 0$	$x = \pm 1$ are saddle points
6-8. $\left((-b - \frac{\partial p}{\partial y})^2 - 4\frac{\partial p}{\partial x}\right)\|_{x=\pm 2,x=0} \geq 0$	$x = \pm 2$ and $x = 0$ are spirals or centers
9-12. $p(x, \dot{x})\|_{x=0,x=\pm 1,x=\pm 2} = 0$	$x = \pm 2$, $x = \pm 1$, and $x = 0$ are fixed points

Notice that $p(x) = (x - 2)(x - 1)x(x + 1)(x + 2)$ with $|b| \leq 4$ satisfies the twelve restrictions. With this p and with $b = -0.1$, the two-dimensional system looks like

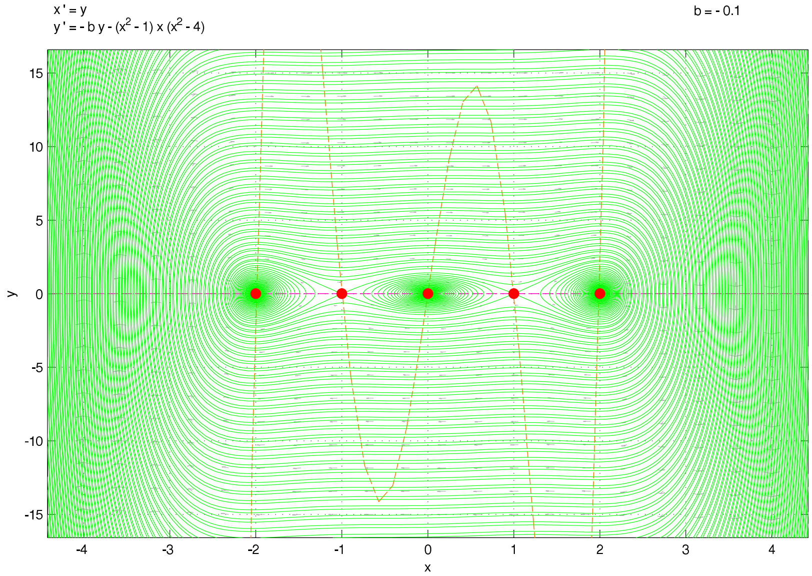

Figure 2: D_3 in the xy-plane with $b = -0.1$ Note that the orange line is the y-nullcline and the green lines are sample orbits.

4 Chaos in the Duffing Family

The extended Duffing equation D_3 exhibits chaos when $\Gamma \neq 0$. The equilibria of any Duffing Family system exist at the x values that satisfy $\Gamma \cos(\omega t) + p(x, 0) = 0$. When $\Gamma \cos(\omega t) \neq 0$, the exact values can be very difficult to solve for any time t. In fact, as time passes, the values of x appear to be unpredictable.

4.1 Bifurcation Diagrams in Γ

A bifurcation diagram is a good indicator that chaos is present in a dynamical system. This type of plot shows how the equilibria change as a set of parameters change in the system. We choose to let Γ vary in D_3 and plot the distance of each equilibrium from the origin. Figure 3 shows that as Γ changes, two equilibria slowly diverge from each other, but at around $\Gamma = 3.2$, more equilibria are created. This is called period doubling. The system becomes exceedingly chaotic at around $\Gamma = 4.5$.

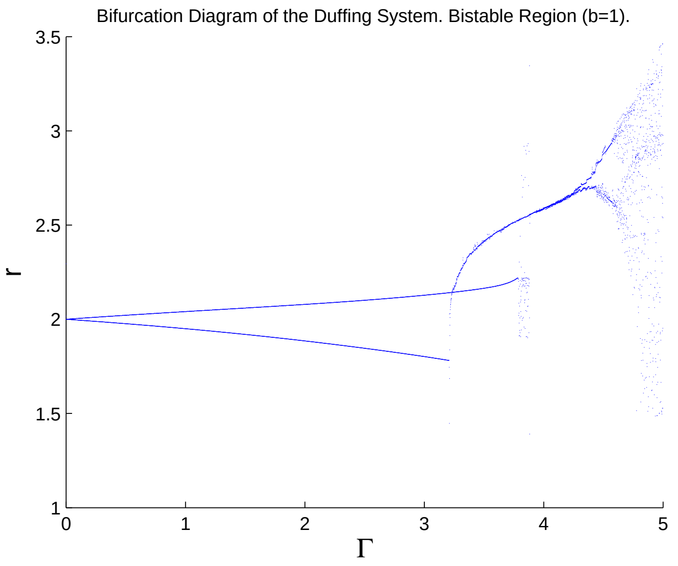

Figure 3: Bifurcation Diagram for D_3 with $b = 1$

If we extend Figure 3 to vary Γ up to 10, we can see that the excessive chaos near $\Gamma = 4.5$ calms down after approximately $\Gamma = 6.1$, but then picks up again at approximately $\Gamma = 8$.

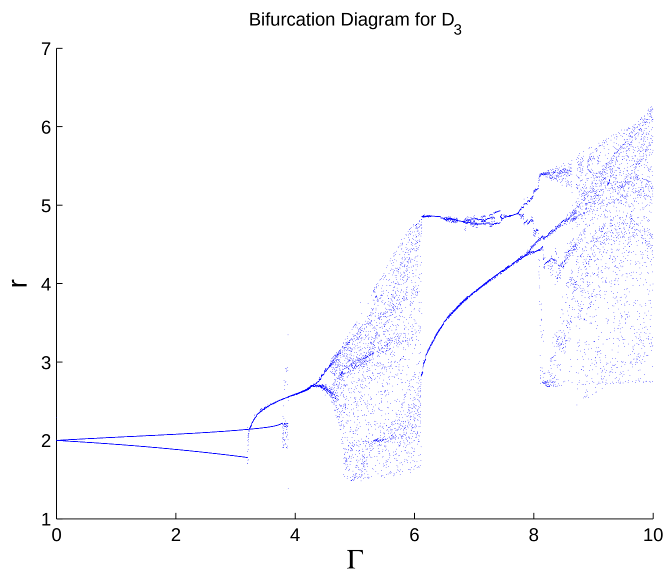

Figure 4: Bifurcation Diagram for D_3 with $b = 1$

For smaller b values, chaos is more apparent. As shown in Figure 5, period doubling occurs when Γ is smaller than 1.

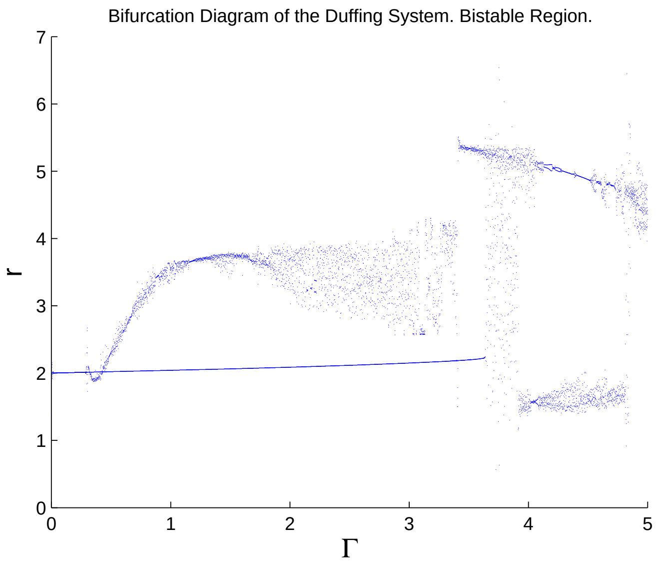

Figure 5: Bifurcation Diagram for D_3 with $b = 0.1$

5 Topologically Conjugate maps

In dynamics, an illuminated level of understanding about a system is revealed when investigating a topologically conjugate map. We take a brief moment to introduce shift maps and then conjecture on their topological relation to the Duffing Family.

5.1 Shift Maps

Let $A_n = \{(a_1, a_2, \ldots, a_k, \ldots) \mid a_i \in \{1, \ldots, n\}\}$ be the set of all sequences whose elements are integers from 1 to n. So, sequences such as $1, 3, 3, 3, 2, ...$ are elements of A_3.

Now, let $\mathbf{T} : A_n \to A_n$ define a mapping by

$$\mathbf{T}(x) \text{ deletes first value of } x \qquad x \in A_n$$

$$\mathbf{T}((a_1, a_2, a_3, \ldots, a_k, \ldots)) = (a_2, a_3, a_4, \ldots)$$

For example $\mathbf{T} : 1, 3, 3, 3, 2, ... \quad \to \quad 3, 3, 3, 2, ...$

5.2 A Poincare Map associated with a Duffing Equation

To create a Poincare Map, one first intersects the state space with a lower-dimensional subspace called a Poincare section. A Poincare Map is a discrete-time dynamical system created by considering the intersection points of the subspace and an orbit given from the equation. The Duffing Equation D_n lies in the plane. So, an appropriate Poincare section would be the line $y = 0$. Instead of creating a Poincare Map from viewing the intersections of an orbit with $y = 0$, we discretize a solution by considering if an orbit spirals completely around a fixed point.
For example, consider an orbit from the D_3 Duffing Equation.

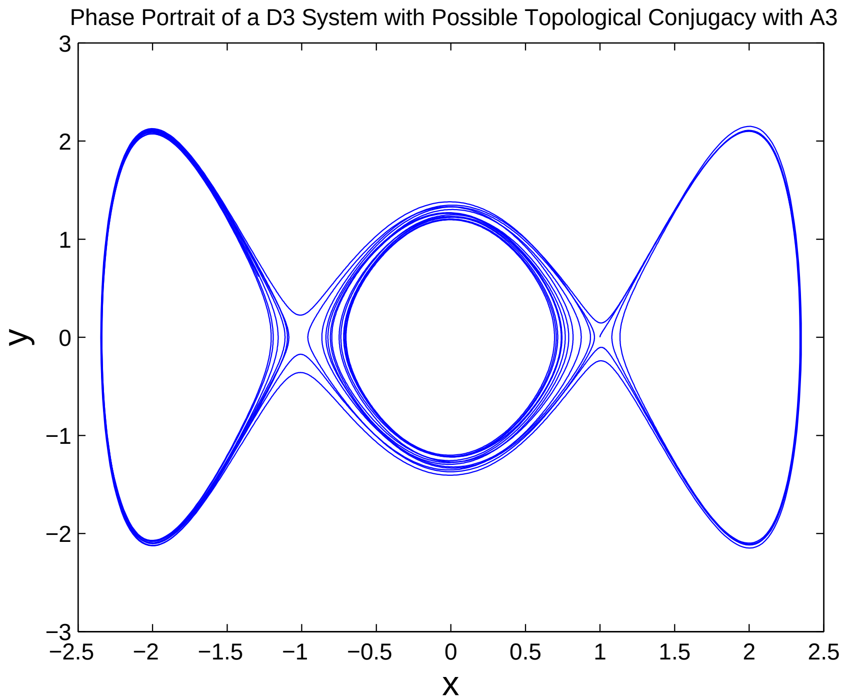

Figure 6: Note the trajectory completely spirals around $\vec{x_2}$ then $\vec{x_3}$ but not $\vec{x_1}$. ($b = .01$,$\Gamma = -.06$)

Label the trajectories that circle around the non-saddle fixed points $x = -2$, $x = 0$, and $x = 2$ as $\vec{x_1}, \vec{x_2}, \vec{x_3}$, respectively. Define a sequence of integers with values $1, 2, 3$ each corresponding to a full rotation around $\vec{x_1}, \vec{x_2}, \vec{x_3}$, respectively. For example, the sequence in Figure 6 is 3,3,1,1,2,2,3. . . .

With this idea that a full spiral is our correspondence map, we have reason to make claim for topological equivalence.

Conjecture: A Poincare Map associated with the D_n Duffing equation is topologically conjugate to the shift map $\mathbf{T}$ on A_n

5.3 Problems with Topological Conjugacy

The D_n maps may contain some problematic parameters that would disqualify them from being topologically equivalent to A_n. First, if the $\vec{x_i}$'s are stable spirals, then a trajectory in the basin of attraction will never spiral around another fixed point. However, if a fixed point exhibits weak attraction properties, that is, it is almost a center, a large enough perturbation amplitude (Γ) may remove a trajectory from the fixed point's basin. For example, as in Figure 7, the phase portrait for the D_3 map with stable spirals has a large enough Γ that removes a trajectory from $\vec{x_3}$'s basin, but is not strong enough to remove it from $\vec{x_2}$'s basin. So, it corresponds to the sequence 3,2,2,2,2,2,. . . .

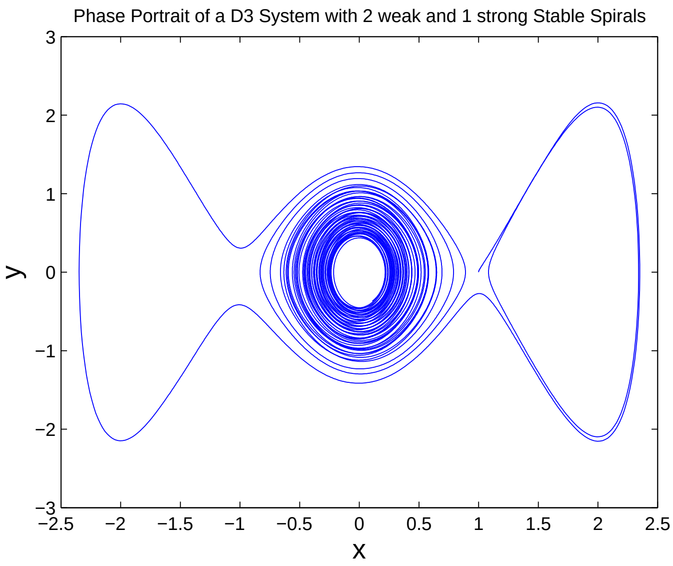

Figure 7: Note the initial trajectory spirals in $\vec{x_3}$'s basin which has weak attraction properties ($b = .0105$). Then a large enough perturbation ($\Gamma = .06$) moves the trajectory into $\vec{x_2}$'s basin where it stays for the remaining time. The corresponding sequence is 3,2,2,2,2,... which prohibits it from being topologically equivalent with the shift map on 3 indices.

Another problem might occur if the map has horribly unstable spirals for fixed points. If this is the case, eventually, trajectories will never fully circle around a fixed point, stopping the sequence. For example, consider the D_3 map in Figure 8. We note that instead of corresponding to an infinite sequence, we have a finite sequence 3,3,3.

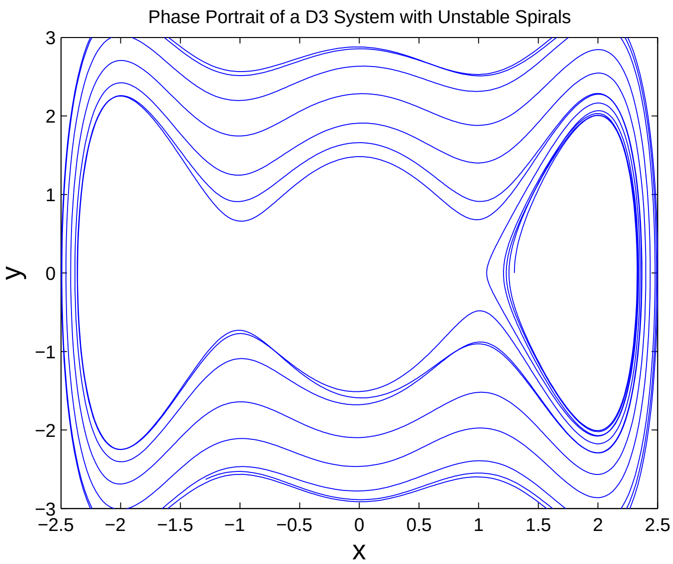

Figure 8: Note the instability of the fixed points ($b = -.01, \Gamma = .1$) keeps the trajectory from ever completing a full circle. This trajectory corresponds to the finite sequence is 3,3,3, which prohibits it from being topologically equivalent with the A_3, whose elements are infinite sequences.

6 Duffing Equation with n Spirals

Making the assumption that $p(x, y)$ is a polynomial in x only, we can continue adding roots to this polynomial to make the Duffing Family infinite. We will classify the Duffing equation in the set based on the number of spirals.

For D_n, let $m = 2n - 1$. This denotes the number of fixed points. Remember that between every spiral fixed point, there must lie a saddle fixed point. (So, for D_2 there are $m = 3$ fixed points; for D_3, $m = 5$ fixed points.)

First, if $p(x, y) = p(x)$, then the Jacobian of the D_n system is:

$$\mathrm{J}f(x, y) = \begin{pmatrix} 0 & 1 \\ -\frac{\partial p}{\partial x} & -b \end{pmatrix} \tag{4}$$

For the fixed points of $x = 0$, $x = \pm 1$, $x = \pm 2$, $\ldots x = \pm m$, we construct the simplest polynomial by the fundamental theorem of algebra:

$$p(x) = x(x^2 - 1^2)(x^2 - 2^2)\ldots(x^2 - m^2)$$

. We require the outermost fixed points $x = \pm m$ to be spirals or centers, the next innermost two to be saddles, and, working inward, alternating the pattern. That is, the set of points $x = \{\pm m, \pm(m-2), \pm(m-4), ...\}$ are spirals or centers, and the set of points $x = \{\pm(m-1), \pm(m-3), \pm(m-5), ...\}$ are saddles. The usual restrictions follow.

Restriction	Reason
1. $\frac{\partial p}{\partial x}\|_{x=\pm m,\ldots} > 0$	$x = \pm 2$, etc. are not saddle points
2. $\frac{\partial p}{\partial x}\|_{x=\pm(m-1),\ldots} < 0$	$x = \pm 1$, etc. are saddle points
3. $\left((-b - \frac{\partial p}{\partial y})^2 - 4\frac{\partial p}{\partial x}\right)\|_{x=\pm m,\ldots} \geq 0$	$x = \pm m$, etc. are spirals or centers
4. $p(x, \dot{x})\|_{x=0,x=\pm 1,\ldots x=\pm m} = 0$	$x = 0$, ..., $x = \pm m$ are fixed points

It should be investigated whether these conditions are met for the D_n equation in general. We believe that there are parameter values of b and Γ such that chaos exists in D_n and that D_n is topologically conjugate to A_n.