Dynamics at the Horsetooth Volume 2, 2010.

On using numerical algebraic geometry to find Lyapunov functions of polynomial dynamical systems

Eric Hanson
Department of Mathematics
Colorado State University hanson@math.colostate.edu

Report submitted to Prof. P. Shipman for Math 540, Fall 2010

Abstract. Lyapunov functions are useful in analyzing the phase plane of dynamical systems. In this paper, we explore the use of techniques from numerical algebraic geometry to find a Lyapunov function of a polynomial dynamical system with a stable fixed point at the origin.

Keywords: Lyapunov function, polynomial dynamical system, numerical algebraic geometry

1 Introduction

Consider systems of the form

$$\dot{x} = f(x), x \in \mathbb{R}^n \tag{1}$$

where $f(x)$ is a system of polynomial equations, for which the trivial solution satisfies $f(x) = 0$.

We take the following as definitions and theorems from [3]:

Definition 1 *For the equation 1 and a neighborhood $D \subset \mathbb{R}^n$ of $x = 0$ let solution for initial condition x_0 be indicated by $x(x_0)$. The solution $x = 0$ is called stable in the sense of Lyapunov (or Lyapunov-stable) if for each $\epsilon > 0$ a $\delta(\epsilon)$ can be found such that $||x_0|| < \delta(\epsilon)$ implies $||x(x_0)|| < \epsilon$.*

Let $V(x)$ be defined and continuously differentiable in $D \subset \mathbb{R}^n$ and $V(0) = 0$

Definition 2 *The function $V(x)$ is called positively (negatively) definite in D if $V(x) > 0$ (< 0) for $x \in D$, $x \neq 0$.*

Definition 3 *The function $V(x)$ is called positively (negatively) semidefinite in D if $V(x) >\geq 0$ (≤ 0) for $x \in D$, $x \neq 0$.*

Definition 4 *The orbital derivative L_t of the function $V(x)$ in the direction of the vectorfield $f(x)$, where x is a solution of equation 1, is*

$$L_t V = \frac{dV}{dx}\dot{x} = \frac{dV}{dx}f(x) = \frac{dV}{dx_1}f_1(x) + \ldots + \frac{dV}{dx_n}f_n(x) \tag{2}$$

Theorem 5 *Consider equation 1. If a function $V(x)$ defined in a neighborhood of $x = 0$ is positively definite with orbital derivative negatively semidefinite, the solution $x = 0$ is stable in the sense of Lyapunov.*

Based on theorem 5, for the system given by equation 1 with a stable equilibrium at $x = 0$ we call $V(x)$ which satisfies the hypotheses of Theorem 5 a Lyapunov function for equation 1.

2 A known example

Consider the following problem that appears in [3]

Problem 6 *Determine the stability of the trivial solution of*

$$\dot{x} = xy^2 - \frac{1}{2}x^3, \dot{y} = -\frac{1}{2}y^3 + \frac{1}{5}x^2y \tag{3}$$

Solution by Verhulst:
$V(x,y) = 1x^2 + 2y^2$ *is a Lyapunov function, thus by Theorem 5 the trivial solution is stable.*

2.1 A symbolic method to find a solution

First we conjecture that V is of the form $V(x,y) = ax^2 + by^2$ where a, b are non-negative real numbers. Then clearly V is defined and differentiable on all of $\mathbb{R}^2$, $V(0,0) = 0$ and $V(x,y) > 0$ for all x, y. So all that remains to find a Lyapunov function is to choose a, b such that L_tV is negatively semidefinite.

$$L_tV = x^2y^2(2a + \frac{2}{5}b) - (ax^4 + by^4)$$

If we consider y as a coefficient on x then the following are solutions to the 4th degree polynomial in x, $V(x) = 0$:

$$x = \pm\frac{1}{5}\sqrt{\frac{25a + 5b \pm 5\sqrt{25a^2 - 15ab + b^2}}{a}} \cdot y$$

Clearly $(0,0)$ is a solution to $L_tV(x,y) = 0$. In $\mathbb{R}^3$ L_tV is a surface and since L_tV is a continuous, differentiable function it can only change from positive to negative in real z if the surface intersects the $z = 0$ plane in a curve or surface. In this way we can force L_tV to be semidefinite in real space by forcing all other roots of L_tV to occur in complex space. In other words the surface defined by L_tV in $\mathbb{R}^3$ intersects the $z = 0$ plane in only points (in this case only the origin).

In this example, we can force all other roots to be complex in their x-coordinate by choosing a, b such that

$$25a^2 - 15ab + b^2 \tag{4}$$

is negative. Then for any $y \neq 0$ the roots of L_tV are in complex space. Solving equation 4 for a:

$$a = \frac{1}{5}(\frac{3}{2} \pm \frac{1}{2}\sqrt{5})b \tag{5}$$

For example, if $b = 1$ then $a = .523607$ and $a = .076393$ are roots to equation 4. Thus any $a \in (.523607, .076393)$ with $b = 1$ will force equation 4 to be negative and thus all the roots of L_tV other than $(0,0)$ to occur in complex space. So the function

$$L_tV(x,y) = x^2y^2(2(0.2) + \frac{2}{5}(1)) - ((0.2)x^4 + (1)y^4) \tag{6}$$

will not change sign in real space. Choosing a test point we see $L_tV(0,1) = -1 < 0$, thus LtV is negative semidefinite for all of $\mathbb{R}^2$. Thus we have constructed $V(x,y) = .2x^2 + y^2$ which is a Lyapunov function of the equation 3

2.2 Limitations of this symbolic approach

This symbolic approach relies on the ability to find an explicit formula for the roots of the orbital derivative. This limits the application of this approach to finding Lyapunov functions for systems where both the candidate Lyapunov function and $\dot{x} = f(x)$ are such that the orbital derivative is either

1. of small enough degree, or
2. a special case of a higher degree polynomial,

so that the quadratic formula, cubic formula, or a method of factoring can be applied to identify the real roots.

Even though this symbolic method is limited to a small class of problems, it can provide an important set of examples for developing numerical methods of finding Lyapunov functions. In the following sections, we try various numerical approaches for finding a Lyapunov function for problem 3. While none of our attempts where successful each provided important insight for developing such numerical techniques.

3 Finding Lyapunov functions numerically

In this section, we explore finding Lyapunov functions using techniques in numerical algebraic geometry. We make two attempts without success, but with important observations for future attempts in numerically finding Lyapunov functions.

3.1 A first attempt

First we must observe that Lyapunov functions are local, that is we only need to satisfy the conditions in a neighborhood of $(0,0)$. We start with the same system solved in section 2.1 and the same candidate Lyapunov function $V(x,y) = ax^2 + by^2$. So to meet the hypotheses of Theorem 5 we again need to find a, b such that L_tV is negatively semidefinite. We attempt to numerically find an a, b that force all roots of L_tV (except $(0,0)$) to be complex in x for $y \in (-1,1)$ again forcing the semidefiniteness of L_tV.

To do this we sample n parings of various a and b values. Then for each paring a, b we take a sample of m values $y \in (-1,1)$ and solve the resulting $L_tV(x) = 0$. Then check that for each y the solutions in x are complex. If a pairing a, b has complex solutions in x for each y in the sample we conjecture that it is a Lyapunov function.

The process described was automated using software developed by Dan Brake and Matt Niemerg. The software automates the sampling process and the process of solving $L_tV(x) = 0$ by calling the software package Bertini.

For this example we found a range of a, b values that we conjectured would yield Lyapunov functions. One such pairing was $a = 1.5, b = 2$, however the resulting

$$L_tV = x^2y^2(2 * 1.5 + \frac{2}{5} * 2) - (1.5 * x^4 - 2y^4) \quad (7)$$

is not negative semidefinite. Solving for x we find four lines

$$x = \pm 0.863703279453978y, x = \pm 1.33691808963518$$

where the surface defined by L_tV passes through the $z = 0$ plane in real space resulting in regions where L_tV is positive.

The problem with this simple approach is that in sampling the y values for a particular a, b we are unable to detect the measure zero set of y values in real space (in this case the four lines) that defines the real roots of L_tV.

3.2 A second attempt

In a second attempt we try to avoid these measure zero issues by simultaneously finding two Lyapunov functions. We start with the same system solved in section 2.1, but consider two candidate Lyapunov functions

$$V_1(x, y) = ax^2 + by^2 \text{ and } V_2(x, y) = cx^2 + dy^2$$

Again we try to choose a, b, c, d such that L_tV_1 and L_tV_2 have roots away from a neighborhood of $(0, 0)$, so that L_tV_1 and L_tV_2 are negative semidefinite.

Again we use the software to automate the processing of sampling a, b, c, d. This approach fails before any conjecture can be made because of the singularity of the root $(0, 0)$ which causes issues with the parameter homotopy used to solve the system of equations in Bertini. However, we note that this singularity at $(0, 0)$ could be used in creating a Lyapunov, similar to how we made use of the relationship between the real numbers and the complex numbers in the symbolic approach of section 2.1. Making use of this singularity could be the key to constructing a Lyapunov function.

4 Conclusion

While these experimental approaches to constructing Lyapunov functions failed, each attempt provided insight to the structure of such functions for polynomial dynamical systems. In these attempts to construct Lypunov functions, we tried to exploit the relationship between the real and complex numbers. Future attempts might benefit from including this approach, but might also consider making use of the singularity of the point $(0, 0)$.

References

[1] D.J. Bates, J.D. Hauenstein, A.J. Sommese, and C.W. Wampler *Bertini: Software for Numerical Algebraic Geometry*, http://www.nd.edu/ sommese/bertini/index.html

[2] D. Brake and M. Niemerg, Software in development.

[3] F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems,* 2nd Edition, Springer-Verlag Berlin Heidelberg, New York, (1990).