

An Application of Perturbation Methods in Evolutionary Ecology

Yang Zou

Department of Mathematics
Colorado State University

`zou@math.colostate.edu`

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Abstract. In this report we apply regular perturbation method to solve a predator-prey model from evolutionary ecology, which is a fast-slow dynamical system. Expansions from regular perturbation method and the reduced system are given, and numerical results are present, which implies this method works very well within small time interval, but diverges for large time. This shows that over large time intervals we deal with a singular perturbation problem, that requires the use of the two-timing method

Keywords: Evolutionary Ecology, Fast-slow dynamics, Regular perturbation method

1 Introduction

Evolutionary ecology lies at the intersection of ecology and evolutionary biology. Ecological and evolutionary changes occur simultaneously and interact with each other. On one hand, ecological changes depend on the observable traits of the population, which are directly associated with some specific genes. On the other hand, evolutionary changes, which are mainly driven by natural selection, will change the frequencies of the associated genes in a population from generation to generation. However, ecological and evolutionary changes have different time scales. Ecological changes often occur faster than evolutionary changes. The above phenomena are also mathematically interesting, as the dynamics can be modeled with help of a coupling fast-slow dynamical system.

To analyze the dynamics on the two different time scales, The traditional method to analyze the dynamics on the two different time scales is a separation of timescales. For the fast system, it is assumed that the slow variable is a constant parameter of the fast system and attractors of the fast system can be found as a function of the slow variable. Then the stable equilibrium value of the fast system is substituted for the slow system. Finally, a dynamical systems analysis is performed on the slow subsystem.

In this project we will explore such fast-slow dynamics by regular perturbation methods. In section 2, we will introduce a simple predator prey model from [1]. Its theoretical and numerical analysis can also be found in [3] and [4]. Section 3 presents application of regular perturbation method on the above model. Numerical methods are presents in section 4, which also includes discussion.

2 A predator prey model

In this paper we apply the regular perturbation method on a simple predator prey model from [1]

$$\begin{aligned}\frac{dx}{dt} &= x(-1 + rx - x^2 - \alpha y), \\ \frac{dy}{dt} &= y[-d - y + (\alpha + 1)x], \\ \frac{d\alpha}{dt} &= \sigma_g^2 x,\end{aligned}\tag{1}$$

where x is the prey population size, y is the predator population size, α is the mean attack rate of the predator, r is a measure of the positive effects of density-dependent interactions in the prey, d is a measure of predator mortality rate, and σ_g^2 is the additive genetic variance in attack rate.

In this paper we choose $r = 3.6$, $d = 0.1$, $\sigma_g^2 = 0.001$; x and y are fast dynamic variables, and α is the slow dynamics variable. Initial conditions for x , y , and α are

$$[x(0), y(0), \alpha(0)] = [1.5, 2.7, 0.8].$$

3 Regular perturbation method

By choosing $\epsilon = \sigma^2 = 0.001$, regular perturbation expansions for x , y , and α are the following

$$\begin{aligned}x &= x_0 + \epsilon x_1 + \dots, \\ y &= y_0 + \epsilon y_1 + \dots, \\ \alpha &= \alpha_0 + \epsilon \alpha_1 + \dots.\end{aligned}\tag{2}$$

As $\epsilon \rightarrow 0$, by plugging (2) into system (1), we get

$$\begin{aligned}\dot{x}_0 + \epsilon \dot{x}_1 + \dots &= (x_0 + \epsilon x_1 + \dots)[-1 + r(x_0 + \epsilon x_1 + \dots) \\ &\quad - (x_0 + \epsilon x_1 + \dots)^2 - (\alpha_0 + \epsilon \alpha_1 + \dots)(y_0 + \epsilon y_1 + \dots)], \\ \dot{y}_0 + \epsilon \dot{y}_1 + \dots &= (y_0 + \epsilon y_1 + \dots)[-d - (y_0 + \epsilon y_1 + \dots) \\ &\quad + (\alpha_0 + \epsilon \alpha_1 + \dots + 1)(x_0 + \epsilon x_1 + \dots)], \\ \dot{\alpha}_0 + \epsilon \dot{\alpha}_1 + \dots &= \epsilon(x_0 + \epsilon x_1 + \dots).\end{aligned}$$

By combining the same order terms and truncating expansions up to $O(\epsilon)$, At the order $O(1)$, we have

$$\begin{aligned}\dot{x}_0 &= x_0(-1 + rx_0 - x_0^2 - \alpha_0 y_0), \\ \dot{y}_0 &= y_0[-d - y_0 + (\alpha_0 + 1)x_0], \\ \dot{\alpha}_0 &= 0.\end{aligned}\tag{3}$$

This result justifies, in fact, the dynamical system approach.

At $O(\epsilon)$, we have

$$\begin{aligned}\dot{x}_1 &= x_1(-1 + rx_0 - x_0^2 - \alpha_0 y_0) + x_0[rx_1 - 2x_0 x_1 - (\alpha_1 y_0 + \alpha_0 y_1)], \\ \dot{y}_1 &= y_1[-d - y_0 + (\alpha_0 + 1)x_0] + y_0(-y_1 + \alpha_0 x_1 + \alpha_1 x_0), \\ \dot{\alpha}_1 &= x_0.\end{aligned}\tag{4}$$

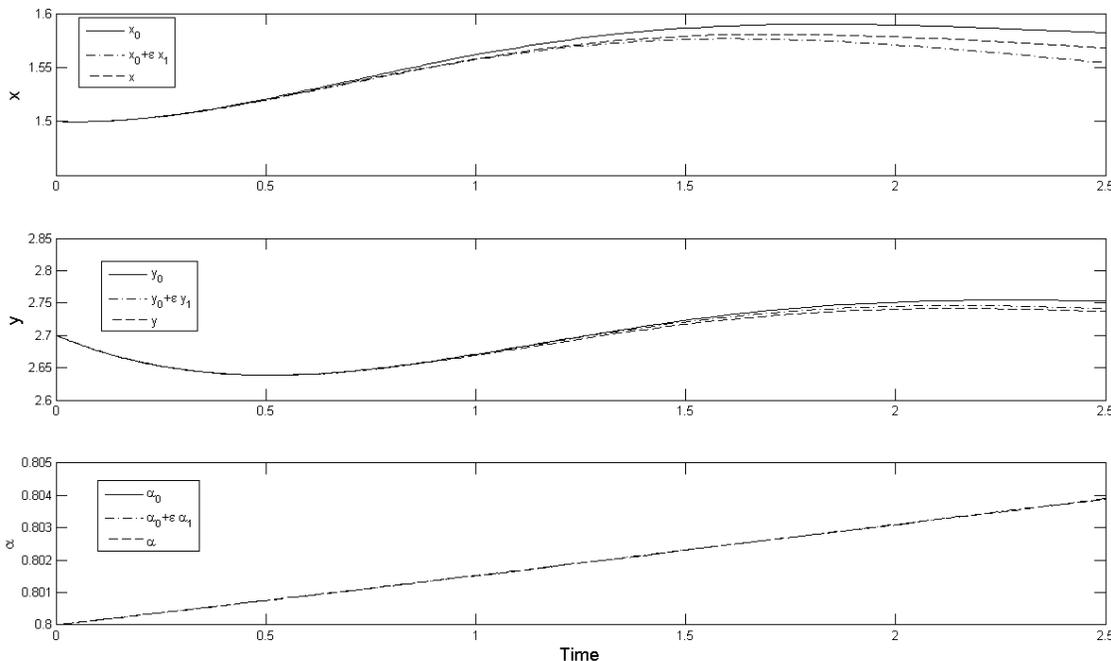


Figure 1: Simulations for $0 \leq t \leq 2.5$. x_0 , y_0 , and α_0 : approximations up to $O(1)$ order. $x_0 + \epsilon x_1$, $y_0 + \epsilon y_1$, and $\alpha_0 + \epsilon \alpha_1$: approximations up to $O(\epsilon)$ order. x , y , and α : exact numerical solutions. Simulations imply good approximations for small t , such as $t < 1$.

Incorporating initial conditions, we have

$$\begin{aligned} [x_0(0), y_0(0), \alpha_0(0)] &= [1.5, 2.7, 0.8], \\ [x_1(0), y_1(0), \alpha_1(0)] &= [0, 0, 0]. \end{aligned}$$

We remark that the rate of change of constant mode in slow variable, α_0 , is equal to zero. The first order mode of the slow variable is related to the constant mode in the fast variable x_0 . The convergence of expansions is proved by the following theorem.

Theorem 1 (Poincare) *Consider the initial value problem*

$$\dot{y} = F(t, y, \epsilon), y(t_0) = \mu$$

with $|t - t_0| \leq h$, $y \in D \subset R^n$, $0 \leq \epsilon \leq \epsilon_0$, $0 \leq \mu \leq \mu_0$. If $F(t, y, \epsilon)$ is continuous with respect to t , y and ϵ and can be expanded in a convergent power series with respect to y and ϵ for $\|y\| \leq \rho$, $0 \leq \epsilon \leq \epsilon_0$, then $y(t)$ can be expanded in a convergent power series with respect to ϵ and μ in a neighborhood of $\epsilon = \mu = 0$, convergent on the time-scale 1.

4 Numerical results and discussions

Simulations are present in Figure (1) and Figure (2). The $O(1)$ orders are labeled as x_0 , y_0 , and α_0 . The expansions up to $O(\epsilon)$ are labeled as $x_0 + \epsilon x_1$, $y_0 + \epsilon y_1$, and $\alpha_0 + \epsilon \alpha_1$. The exact numerical solutions are labeled as x , y , and α . These two figures imply that regular perturbation method works pretty good for small t , such as $t < 1$. But for big t , such as $t = 5$, the errors are big.

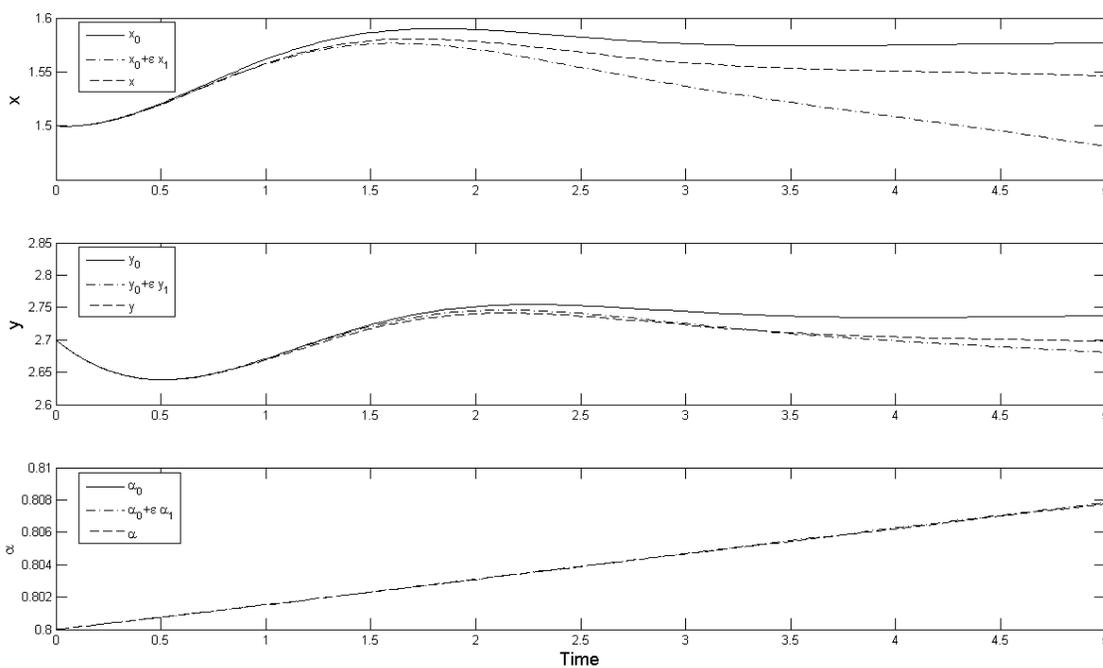


Figure 2: Simulations for $0 \leq t \leq 5$. x_0 , y_0 , and α_0 : approximations up to $O(1)$ order. $x_0 + \epsilon x_1$, $y_0 + \epsilon y_1$, and $\alpha_0 + \epsilon \alpha_1$: approximations up to $O(\epsilon)$ order. x , y , and α : exact numerical solutions. Simulations imply poor approximations for big t , such as $t = 5$.

Improvements can be made by using two-timing method. To apply two-timing to system (1), let $\tau = t$ denote the fast time, and let $T = \epsilon^\gamma$ denote the slow time. These two times are treated as if they were independent variables. Then the solutions of system (1) are expanded as series

$$\begin{aligned}x(t, \epsilon) &= x_0(\tau, T) + \epsilon^{\theta_1} x_1(\tau, T) + \cdots, \\y(t, \epsilon) &= y_0(\tau, T) + \epsilon^{\theta_2} y_1(\tau, T) + \cdots, \\\alpha(t, \epsilon) &= \alpha_0(\tau, T) + \epsilon^{\theta_3} \alpha_1(\tau, T) + \cdots,\end{aligned}$$

where γ , θ_1 , θ_2 , and θ_3 are determined in expansions. One drawback of the above expansions is that the system becomes a system of partial differential equations (PDEs), which are too complicated to get analytical solution like pendulum equations. But if we can manage to solve this system of PDEs numerically, the approximations will work for a longer time range.

References

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