

## An Introduction to Averaging Method

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**Abstract.** As one of the classical methods for analyzing nonlinear oscillations, the averaging method is particularly useful for weakly nonlinear problems. This report introduces its basic idea, and demonstrates its applications in two simple examples.

### 1 Introduction

Averaging method is a useful computational technique. Its basic idea can be dated back to the late 18th century, when, in 1788, Lagrange formulated the gravitational three-body problem as a perturbation of the two-body problem. No rigorous proof of its validity was given, until Fatou gave the first proof of the asymptotic validity of the method in 1928. After the systematic researches done by Krylov, Bogoliubov, Mitropolsky etc, in 1930s, the averaging method gradually became one of the classical methods in analyzing nonlinear oscillations. This report briefly introduces its basic idea, and demonstrates its applications in two examples.

### 2 Basic Idea of the Averaging Method

Averaging method is applicable to systems of the form:

$$\dot{x} = \epsilon f(x, t, \epsilon); \quad x \in U \subseteq \mathbb{R}^n, \quad \epsilon \ll 1 \quad (2.1)$$

where  $f : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$  is  $C^r$ ,  $r \geq 1$ , bounded on bounded sets, and of period  $T > 0$  in  $t$ ;  $U$  is bounded and open. The associated autonomous averaged system is defined as

$$\dot{y} = \epsilon \frac{1}{T} \int_0^T f(y, t, 0) dt \equiv \epsilon \bar{f}(y) \quad (2.2)$$

The basic idea of the averaging method is to approximate the original system (2.1) by the averaged system (2.2), which is presumably easier to study, and infer properties of the dynamics of the original system by the understanding of the dynamics of the averaged system. While in practice, it turns out that this idea works well, more questions arise:

1. in practice, weakly nonlinear system is usually given as:

$$\dot{x} = Ax + \epsilon f(x, t, \epsilon)$$

which is not of the form (2.1). How can the averaging method be applied in these systems?

2. How do qualitative properties of the solutions of the averaged system correspond to those of the original system?

We answer these two questions in next 2 subsections.

### 3 The Lagrange Standard Form

Consider the initial value problem:

$$\dot{x} = A(t)x + \epsilon g(x, t), x(0) = x_0 \quad (3.1)$$

where  $A(t)$  is a continuous  $n \times n$ -matrix,  $g(x, t)$  is a sufficiently smooth function of  $t$  and  $x$ .

Let  $\Phi(t)$  be the fundamental matrix of the unperturbed system ( $\epsilon = 0$ ), and  $y(t)$  be such that  $y(0) = x_0$ . Introducing the comoving coordinates as:

$$x = \Phi(t)y \quad (3.2)$$

then

$$\dot{x} = \dot{\Phi}(t)y(t) + \Phi(t)\dot{y}$$

Since  $x(t)$  is the solution to (3.1), we have

$$\dot{\Phi}(t)y(t) + \Phi(t)\dot{y} = A(t)\Phi(t)y + \epsilon g(\Phi(t)y(t), t)$$

i.e.,

$$\Phi(t)\dot{y} = (A(t)\Phi(t) - \dot{\Phi}(t))y(t) + \epsilon g(\Phi(t)y(t), t).$$

Since  $\Phi(t)$  is the fundamental matrix of the unperturbed system, i.e.,  $\dot{\Phi}(t) = A(t)\Phi(t)$ , we get

$$\Phi(t)\dot{y} = \epsilon g(\Phi(t)y(t), t)$$

i.e.,

$$\dot{y} = \epsilon \Phi^{-1}(t)g(\Phi(t)y(t), t). \quad (3.3)$$

(3.3) is said to have the *Lagrange standard form*, and it is in general written as:

$$\dot{y} = \epsilon f(y, t). \quad (3.4)$$

Thus, without loss of any generality, we discuss the weakly nonlinear differential equations in *Lagrange standard form*, which is exact in the form of (2.1).

## 4 The Averaging Theorem

To answer the second question in subsection 2.1, we state the averaging theorem, and explain how to apply it. The proof can be found in [1, 2, 3, 4].

**Theorem 1. (The Averaging Theorem)** *There exists a  $C^r$  change of coordinates  $x = y + \epsilon w(y, t, \epsilon)$  under which (2.1) becomes*

$$\dot{y} = \epsilon \bar{f}(y) + \epsilon^2 f_1(y, t, \epsilon) \quad (4.1)$$

where  $f_1$  is of period  $T$  in  $t$ . Moreover,

- (i) *If  $x(t)$  and  $y(t)$  are solutions of (2.1) and (2.2) based at  $x_0, y_0$ , respectively, at  $t = 0$ , and  $|x_0 - y_0| = \mathcal{O}(\epsilon)$ , then  $|x(t) - y(t)| = \mathcal{O}(\epsilon)$  on a time scale  $t \sim \frac{1}{\epsilon}$ .*
- (ii) *If  $p_0$  is a hyperbolic fixed point of (2.2) then there exists  $\epsilon_0 > 0$  such that, for all  $0 < \epsilon \leq \epsilon_0$ , (2.1) possesses a unique hyperbolic periodic orbit  $\gamma_\epsilon(t) = p_0 + \mathcal{O}(\epsilon)$  of the same stability type as  $p_0$ .*
- (iii) *If  $x^s(t) \in W^s(\gamma_\epsilon)$  is a solution of (2.1) lying in the stable manifold of the hyperbolic periodic orbit  $\gamma_\epsilon = p_0 + \mathcal{O}(\epsilon)$ ,  $y^s(t) \in W^s(p_0)$  is a solution of (2.2) lying in the stable manifold of the hyperbolic fixed point  $p_0$  and  $|x^s(0) - y^s(0)| = \mathcal{O}(\epsilon)$ , then  $|x^s(t) - y^s(t)| = \mathcal{O}(\epsilon)$  for  $t \in [0, \infty)$ . Similar results apply to solutions lying in the unstable manifolds on the time interval  $t \in (-\infty, 0]$ .*

The averaging theorem provides a way to transform the original system into an asymptotic expansion. Here we show the expansion by explicitly computing the change of coordinates. Let

$$f(x, t, \epsilon) = \bar{f}(x) + \tilde{f}(x, t, \epsilon) \quad (4.2)$$

be split into its mean,  $\bar{f}$ , and oscillating part  $\tilde{f}$ . Since as described in the theorem,  $x = y + \epsilon w(y, t, \epsilon)$ , differentiating both sides, we get:

$$\dot{x} = \dot{y} + \epsilon D_y w(y, t, \epsilon) \dot{y} + \epsilon \frac{\partial w}{\partial t}.$$

Since

$$\dot{x} = \epsilon(\bar{f}(x) + \tilde{f}(x, t, \epsilon))$$

we have

$$\epsilon(\bar{f}(x) + \tilde{f}(x, t, \epsilon)) = (I + \epsilon D_y w(y, t, \epsilon)) \dot{y} + \epsilon \frac{\partial w}{\partial t}$$

Thus,

$$\dot{y} = \epsilon(I + \epsilon D_y w(y, t, \epsilon))^{-1}(\bar{f}(x) + \tilde{f}(x, t, \epsilon) - \frac{\partial w}{\partial t})$$

i.e.,

$$\dot{y} = \epsilon(I + \epsilon D_y w(y, t, \epsilon))^{-1}(\bar{f}(y + \epsilon w) + \tilde{f}(y + \epsilon w, t, \epsilon) - \frac{\partial w}{\partial t}). \quad (4.3)$$

Expand the right handside of the above equation (4.3) into power series of  $\epsilon$ , we get:

$$(I + \epsilon D_y w(y, t, \epsilon))^{-1} = I - \epsilon D_y w + \epsilon^2 (D_y w)^2 + \mathcal{O}(\epsilon^3),$$

$$\bar{f}(y + \epsilon w) = \bar{f}(y) + D_y \bar{f}(y) w(y, t, 0) \epsilon + \mathcal{O}(\epsilon^2),$$

and

$$\tilde{f}(y + \epsilon w, t, \epsilon) = \tilde{f}(y, t, 0) + [D_y \tilde{f}(y, t, 0) w(y, t, 0) + \frac{\partial \tilde{f}}{\partial \epsilon}(y, t, 0)] \epsilon + \mathcal{O}(\epsilon^2).$$

Thus,

$$\dot{y} = \epsilon (I - \epsilon D_y w(y, t, 0) + \mathcal{O}(\epsilon^2)) (\bar{f}(y) + \tilde{f}(y, t, 0) + [D_y \tilde{f}(y, t, 0) w(y, t, 0) + \frac{\partial \tilde{f}}{\partial \epsilon}(y, t, 0)] \epsilon - \frac{\partial w}{\partial t} + \mathcal{O}(\epsilon^2)).$$

Take

$$\frac{\partial w}{\partial t} = \tilde{f}(y, t, 0) \tag{4.4}$$

then

$$\dot{y} = \epsilon (I - \epsilon D_y w(y, t, 0) + \mathcal{O}(\epsilon^2)) (\bar{f}(y) + [D_y \tilde{f}(y, t, 0) w(y, t, 0) + \frac{\partial \tilde{f}}{\partial \epsilon}(y, t, 0)] \epsilon + \mathcal{O}(\epsilon^2)),$$

i.e.,

$$\dot{y} = \epsilon (\bar{f}(y) + [D_y \tilde{f}(y, t, 0) w(y, t, 0) + \frac{\partial \tilde{f}}{\partial \epsilon}(y, t, 0) - D_y w(y, t, 0) \bar{f}(y)] \epsilon + \mathcal{O}(\epsilon^2)).$$

Therefore,

$$\dot{y} = \epsilon \bar{f}(y) + \epsilon^2 [D_y \tilde{f}(y, t, 0) w(y, t, 0) + \frac{\partial \tilde{f}}{\partial \epsilon}(y, t, 0) - D_y w(y, t, 0) \bar{f}(y)] + \mathcal{O}(\epsilon^3). \tag{4.5}$$

In addition to providing an approximation of the original system in asymptotic expansion, the averaging theorem also implies that the averaged system (2.2) can be used to approximate stable and unstable manifolds of the original system (2.1) in bounded sets, and to study the global structure of the Poincare map of the original system (2.1). In next section, we demonstrate how to apply the asymptotic expansion (4.5) and check the validity of the conclusions of the theorem in two simple examples.

## 5 Examples

**Example 1.** Consider the scalar system

$$\dot{x} = -\epsilon x \cos t \tag{5.1}$$

Let

$$f(x, t, \epsilon) = -x \cos t,$$

then

$$\bar{f}(x) = -x \frac{1}{2\pi} \int_0^{2\pi} \cos t dt = 0, \text{ and } \tilde{f}(x, t, \epsilon) = -x \cos t.$$

Since

$$\frac{\partial w}{\partial t} = \tilde{f}(y, t, 0)$$

we have that

$$w = \int_0^t (-y) \cos s ds = -y \sin t$$

Substituting  $\bar{f}$ ,  $\tilde{f}$  and  $w$  obtained above into (4.5), we get:

$$\dot{y} = \epsilon^2 [y \cos t \sin t] + \mathcal{O}(\epsilon^3)$$

Neglecting the  $\mathcal{O}(\epsilon^2)$  terms, the autonomous averaged equation is simply

$$\dot{y} = 0$$

The exact solution of the original system is:  $x(t) = x(0)e^{-\epsilon \sin(t)}$ , and the solution of the approximated system is:  $y(t) = y(0)$ . Choose  $x(0) = x_0$  and  $y(0) = y_0$  such that  $|x_0 - y_0| = \mathcal{O}(\epsilon)$ , then  $|x(t) - y(t)| = |x_0 e^{-\epsilon \sin(t)} - y_0|$ . Expanding  $e^{-\epsilon \sin(t)}$  around  $\epsilon = 0$  gives:

$$e^{-\epsilon \sin(t)} = 1 - \epsilon \sin(t) + \frac{\sin^2(t)}{2} \epsilon^2 + \mathcal{O}(\epsilon^3).$$

Thus,

$$|x(t) - y(t)| = |x_0 - y_0 - \epsilon \sin(t) + \frac{\sin^2(t)}{2} \epsilon^2 + \mathcal{O}(\epsilon^3)| \leq |x_0 - y_0| + |\epsilon \sin(t) + \frac{\sin^2(t)}{2} \epsilon^2 + \mathcal{O}(\epsilon^3)| = \mathcal{O}(\epsilon)$$

Obviously, 0 is the fixed point of both systems, and both systems have no hyperbolic limit set.

**Example 2.** Consider the scalar system

$$\dot{x} = \epsilon(-x + \cos^2 t) \tag{5.2}$$

Let  $f(x, t, \epsilon) = -x + \cos^2 t$ , then  $\bar{f}(x) = \frac{1}{\pi} \int_0^\pi (-x + \frac{\cos(2t)+1}{2}) dt = \frac{1}{2} - x$ , and  $\tilde{f}(x, t, \epsilon) = \cos^2 t - \frac{1}{2}$ . Take

$$\frac{\partial w}{\partial t} = \cos^2 t - \frac{1}{2} = \frac{1}{2} \cos 2t,$$

then

$$w(y, t, \epsilon) = \frac{\sin(2t)}{4} \tag{5.3}$$

Substituting  $\bar{f}$ ,  $\tilde{f}$  and  $w$  obtained above into (4.5), we get the transformed equation:

$$\dot{y} = \epsilon \left( \frac{1}{2} - y \right) - \epsilon^2 \frac{\sin 2t}{4} + \mathcal{O}(\epsilon^3)$$

So the autonomous averaged equation is:

$$\dot{y} = \epsilon\left(\frac{1}{2} - y\right) \quad (5.4)$$

and its exact solution is:

$$y(t) = \left(y_0 - \frac{1}{2}\right)e^{-\epsilon t} + \frac{1}{2} \quad (5.5)$$

where  $y_0 = y(0)$ .

The exact solution of (5.2) is:

$$x(t) = \left(x_0 - \frac{\epsilon^2 + 2}{\epsilon^2 + 4}\right)e^{-\epsilon t} + \frac{\epsilon^2 \cos(2t) + 2\epsilon \sin(2t) + 4 + \epsilon^2}{2(\epsilon^2 + 4)} \quad (5.6)$$

where  $x_0 = x(0)$ .

Comparing this with the approximated solution (5.5), we get:

$$|x(t) - y(t)| = \left| \left(x_0 - y_0\right)e^{-\epsilon t} + \left(\frac{1}{2} - \frac{\epsilon^2 + 2}{\epsilon^2 + 4}\right)e^{-\epsilon t} + \frac{\epsilon^2 \cos(2t) + 2\epsilon \sin(2t)}{2(\epsilon^2 + 4)} \right|$$

i.e.,

$$|x(t) - y(t)| = \left| \left(x_0 - y_0\right)e^{-\epsilon t} - \frac{\epsilon^2}{2(\epsilon^2 + 4)}e^{-\epsilon t} + \frac{\cos(2t)}{2(\epsilon^2 + 4)}\epsilon^2 + \frac{\sin(2t)}{(\epsilon^2 + 4)}\epsilon \right|$$

Expand  $e^{-\epsilon t}$  and  $\frac{1}{4+\epsilon^2}$  in above equality to power series of  $\epsilon$ , i.e.,

$$e^{-\epsilon t} = 1 - t\epsilon + \frac{t^2}{2}\epsilon^2 + \mathcal{O}(\epsilon^3)$$

$$\frac{1}{4 + \epsilon^2} = \frac{1}{4} \frac{1}{1 + \frac{\epsilon^2}{4}} = \frac{1}{4} \left(1 - \frac{\epsilon^2}{4} + \mathcal{O}(\epsilon^4)\right)$$

then

$$|x(t) - y(t)| = \left| \left(x_0 - y_0\right)\left(1 - t\epsilon + \mathcal{O}(\epsilon^2)\right) - \frac{\epsilon^2}{8}\left(1 - \frac{\epsilon^2}{4} + \mathcal{O}(\epsilon^4)\right)\left(1 - t\epsilon + \mathcal{O}(\epsilon^2)\right) \right. \\ \left. + \frac{\cos(2t)}{8}\left(1 - \frac{\epsilon^2}{4} + \mathcal{O}(\epsilon^4)\right)\epsilon^2 + \frac{\sin(2t)}{4}\left(1 - \frac{\epsilon^2}{4} + \mathcal{O}(\epsilon^4)\right)\epsilon \right|$$

i.e.,

$$|x(t) - y(t)| = \left| \left(x_0 - y_0\right) - \left(x_0 - y_0\right)t\epsilon + \mathcal{O}(\epsilon^2) + \frac{\sin(2t)}{4}\epsilon + \mathcal{O}(\epsilon^2) \right|$$

Choosing  $|x_0 - y_0| = \mathcal{O}(\epsilon)$ , then it is very easy to see that

$$|x(t) - y(t)| = \left| \left(x_0 - y_0\right) + \frac{\sin(2t)}{4}\epsilon + \mathcal{O}(\epsilon^2) \right| = \mathcal{O}(\epsilon)$$

This is in agreement with conclusion (i) of the theorem. The autonomous averaged system (5.4) has a hyperbolic fixed point:  $y = \frac{1}{2}$ , which is a sink, and corresponds to the attracting periodic orbit in the original system (5.2):  $x_\gamma(t) = \frac{\epsilon^2 \cos(2t) + 2\epsilon \sin(2t) + 4 + \epsilon^2}{2(\epsilon^2 + 4)}$ . Arbitrarily choose an initial value  $x_0$ , it is easy to see that for  $t \rightarrow \infty$ ,  $x(t) \rightarrow x_\gamma(t)$ . Hence the results are in agreement with the conclusions of the theorem.

## 6 Summary

In this report, we have introduced the basic idea of the averaging method. Without the proof we restated the averaging theorem, and showed how to transform a weakly nonlinear differential equation system in nonstandard form to the Lagrange standard form for analyzing. By explaining the theorem, we showed how to derive an approximation of the system in Lagrange standard form. From this approximation in asymptotic series form, the autonomous averaged system can be easily obtained. By demonstrating how to use the averaging method in analyzing the dynamics of two simple examples, we verified conclusions of the theorem.

## References

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