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Three-Wave Interactions of Spin Waves

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Abstract. Spin waves are waves of dynamical magnetization which can be excited in ferromagnetic materials. These waves can be thought of as slow electromagnetic waves. However, electromagnetic waves in a vacuum are described completely by Maxwell's equations, and spin waves are not. To describe the dynamics of spin waves, one needs to consider the Landau-Lifshitz equation in addition to Maxwell's equations. This article will give a heuristic derivation of the Landau-Lifshitz equation, and use it to analyze spin waves in an unbounded medium. An overview of the linear theory of spin waves in an infinite medium will be given, and nonlinear equations that describe the three-wave interaction of spin waves propagating in an infinite medium will be derived from the Landau-Lifshitz equation.

Keywords: Spin Waves, Three-Wave Interactions

1 Introduction

Spin waves are waves of dynamical magnetization that can be excited in ferromagnetic materials which have been magnetized to saturation. These waves are typically analyzed by solving Maxwell's equations in the magnetostatic limit, along with the Landau-Lifshitz equation, which determines how the magnetization of a ferromagnetic material evolves in time. In general, both the linear and nonlinear properties of these waves depend strongly on the geometric configuration of the sample, the applied field, and the direction of propagation of the waves. The linear and nonlinear characteristics of these waves have been heavily studied, and much discussion of these waves can be found in [2] and [3]. Still, the nonlinear characteristics of spin waves are being explored.

Experiments have shown that chaotic spin waves can be excited though three wave interactions in a yttrium iron garnet(YIG) film based feedback ring [1]. In this configuration, the loss in the film is compensated by a microwave amplifier. There are two microstrip antennas placed several millimeters apart on the film. One microstrip antenna excites spin waves and the other detects spin waves. Spin waves are detected by one antenna, the signal from this antenna is amplified by a microwave amplifier, and the output from this amplifier is sent to the other microstrip antenna, which excites spin waves in the film. These waves propagate to the detection antenna, and the cycle starts again.

For the analysis in this paper, the effects of the amplification and the damping experienced by the spin waves will be ignored, in hope that some insight into the behavior of this system can still be gained from the analysis of the conservative system.

There is one detail of the feedback ring configuration that we cannot escape, as it is essential to even the conservative theory. That is waves with certain wavenumbers are able to make a single trip around the ring and experience a phase shift of $2\pi n$. These are called resonant eigenmodes, and it is these modes that are directly driven by the amplifier. These eigenmodes undergo nonlinear three-wave interactions with spin waves at about half of the frequency. These half-frequency spin waves are not ring eigenmodes. These modes are not directly driven by the amplifier; they are only excited though by the decay of an eigenmode. The goal of this analysis is to gain some insight into the way that energy flows between the eigenmodes and the half-frequency spin waves by the three-wave interaction. This exchange of energy is responsible for the chaos observed in [1].

In [1] there are many resonant eigenmodes excited, and they are all undergoing three-wave interactions. This paper will analyze a simpler situation. There will be one linearly excited mode, and it will interact with two half-frequency modes. Also, the boundary effects of the thin-film geometry will be ignored. The waves will be assumed to be propagating in an infinite medium. Note that the half-frequency spin waves involved in this proccess have huge wavenumbers, and so it may not be unreasonable to consider these waves as propagating in an unbounded medium. This is, however, an unreasonable assumtion to make for the ring eigenmode. The inverse of its wavenumber is on the order of the size of the film. Hopefully, the equations that will be derived in this highly idealized situation can still give some insight into the behavior of the feedback ring.

2 Background: The Landau-Lifsitz equaion

The magnetic properties of a material come from the magnetic moments of its electrons. If these magnetic moments are aligned in a particular direction, the material will have a net magnetic moment in that direction. Magnetization is the dipole moment per unit volume, so it is a density of magnetic moment. Magnetization evolves in time in a complicated way, and this is given by the Landau-Lifshitz equation.

The Landau-Lifshitz equation arises from the spin of an electron. Electrons have an intrinsic spin angular momentum, and they also have an intrinsic magnetic moment. These are related by

$$\vec{\mu} = -|\gamma|\vec{L} \tag{1}$$

Here, $\vec{\mu}$ is the electron magnetic moment, $\vec{L}$ is the angular momentum, and $-|\gamma|$ is the electron gyromagnetic ratio. It is written with an absolute value sign to emphasize that it is a negative number. Also, a magnetic moment in an applied field feels a torque.

$$\vec{\tau} = \vec{\mu} \times \vec{H} \tag{2}$$

Torque is the time derivative of momentum.

$$\frac{d\vec{L}}{dt} = \vec{\mu} \times \vec{H} \tag{3}$$

From (1) we have

$$\frac{d\vec{\mu}}{dt} = \ -|\gamma|(\vec{\mu} \times \vec{H}). \tag{4}$$

In a ferromagnetic material, all of the electrons in a small region that contribute to the overall magnetization have magnetic moments that point in the same direction. So, as each electron magnetic moment changes according to the above equation, they all change in precisely the same way. So, (4) describes the evolution of the magnetic moment of a small region of a material as well as the evolution of the magnetic moment of an electron. So, if one divides both sides of the equation by the volume under consideration, and recall that magnetization is dipole moment per unit volume, one finds

$$\frac{d\vec{M}}{dt} = -|\gamma|(\vec{M} \times \vec{H}). \tag{5}$$

Here, $\vec{M}$ is the magnetization. This is the Landau-Lifshitz equation, and it is very important in magnetization dynamics.

3 Incorporation of Maxwell's Equations

Electromagnetic magnetic waves are typically described by Maxwell's equations. Spin waves are more complicated, because the Landau-Lifshitz equation also plays a role. Still, Maxwell's equations are necessary to describe the magnetic interactions between different parts of the material. Ampere's law is one of the two Maxwell's equations that describes magnetism.

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}. \tag{6}$$

Here, $\vec{J}$ is the current density, $\vec{D}$ is the electric displacement field, and t is time. The current density term, $\vec{J}$, is dropped in the standard analysis. This is easy to justify physically. YIG is an insulator, and so there will not be any currents in the sample. Dropping the current density term, (6) becomes

$$\nabla \times \vec{h} = \frac{\partial \vec{D}}{\partial t}. \tag{7}$$

Here h is a dynamical magnetic field associated with the spin waves. It is denoted by a lowercase letter to emphasize that the field associated with the spin waves is small. If one takes the curl of both sides of (7), and uses the vector identity $\nabla \times (\nabla \times \vec{v}) = \nabla(\nabla \cdot \vec{v}) - \nabla^2 \vec{v}$, one finds

$$\nabla(\nabla \cdot \vec{h}) = \nabla^2 \vec{h} + \nabla \times \frac{\partial \vec{D}}{\partial t}. \tag{8}$$

The other one of Maxwell's equations relevant to magnetism is Gauss's law: $\nabla \cdot \vec{b} = 0$. If one considers that $\vec{b} = \mu_0(\vec{h} + \vec{m})$, then Gauss's law becomes

$$\nabla \cdot \vec{h} = -\nabla \cdot \vec{m}. \tag{9}$$

Combining (9) and (8) yields

$$-\nabla(\nabla \cdot \vec{m}) = \nabla^2 \vec{h} + \nabla \times \frac{\partial \vec{D}}{\partial t} \tag{10}$$

or, since partial derivatives commute,

$$-\nabla(\nabla \cdot \vec{m}) = \nabla^2 \vec{h} + \frac{\partial \nabla \times \vec{D}}{\partial t}. \tag{11}$$

If there were an electric displacement field, it would be generated by a changing magnetic field. This process is described by another of Maxwell's equations, Faraday's law, which states

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \tag{12}$$

Note that since YIG is not very polarizable, $\vec{D} \approx \epsilon_0 \vec{E}$. Also note that $b = \mu_0(\vec{h} + \vec{m})$. Finally, note that $\epsilon_0 \mu_0 = 1/c^2$, where c is the speed of light in a vacuum. With these considerations, from (11) and (12) we have

$$-\nabla(\nabla \cdot \vec{m}) = \nabla^2 \vec{h} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\vec{h} + \vec{m}). \tag{13}$$

This is a relationship between the dynamic magnetization and the dynamic magnetic field associated with the spin waves. We will consider the case of a plane spin wave. The wavevectors of the magnetization and the magnetic field can be assumed to be the same, since they are associated with the same wave.

$$\vec{h} = \vec{h}_0 e^{i\vec{k}\cdot\vec{r}} \tag{14}$$

$$\vec{m} = \vec{m}_0 e^{i\vec{k}\cdot\vec{r}} \tag{15}$$

If the magnetic fields and magnetization have this form, it follows from (13) that

$$\vec{h_0} = -\vec{k}\frac{(\vec{m_0} \cdot \vec{k})}{k^2} - \frac{v^2}{c^2}(\vec{h}_0 + \vec{m}_0). \tag{16}$$

Here, $v = \omega/k$ is the phase velocity of the spin waves. This equation can be simplified a great deal. The waves under consideration have a phase velocity signifigantly slower than the speed of light. Typically, v is on the order of .005c for the ring eigenmodes, and the phase velocity of the half-frequency modes is much smaller still. The term $-(v^2/c^2)(\vec{h}_0 + \vec{m}_0)$ is negligibly small, and can be dropped. It is customary to drop this term very early in the analysis by stating early on that $\partial\vec{D}/\partial t = 0$. After making this very accurate approximation, we find

$$\vec{h_0} = -\vec{k}\frac{(\vec{m_0} \cdot \vec{k})}{k^2}. \tag{17}$$

This equation is useful because it will allow us to “plug” Maxwell’s equations into the Landau-Lifshitz equation.

4 Linear spin waves

This section examines the linear properties of spin waves propagating in a infinite medium. In this context, “linear” means that the amplitude of the spin waves is small. If the amplitude of a spin wave mode becomes very large, it can excite other modes, which will interact with it in a complicated way. If the spin wave amplitude remains small, the behavior of the spin waves is much simpler. We will look for plane spin waves. These will correspond to magnetization and magnetic field distributions of the form

$$\vec{h} = \vec{h}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)} \tag{18}$$

$$\vec{m} = \vec{m}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}. \tag{19}$$

We will use the Landau-Lifshitz equation to analyze these waves. Consider an unbounded medium, in a strong applied filed $\vec{H}_0$. The medium will be magnetized to saturation in the same direction. This magnetization will be denoted $\vec{M}_0$. The direction of both $\vec{H}_0$ and $\vec{M}_0$ will be taken to be the $\hat{z}$ direction. There will be a plane wave magnetic field distribution $\vec{h}e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ associated with the spin waves. There will also be a plane wave magnetization distribution $\vec{m}e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ associated with the spin waves. The wavevector $\vec{k}$ will be taken to lie in the yz plane. That is $\vec{k} = k_y\hat{y} + k_z\hat{z}$.

This assumption can be made with no loss of generality, since whatever component of the wavevector that is parallel to the applied field can be called k_z, and whatever component of $\vec{k}$ is perpendicular to the applied field can be called k_y. So, the magnetization and the magnetic field under consideration are given by

$$\vec{H} = H_0\hat{z} + \vec{h}e^{i(\vec{k}\cdot\vec{r})} \tag{20}$$

$$\vec{M} = M_0\hat{z} + \vec{m}e^{i(\vec{k}\cdot\vec{r})}. \tag{21}$$

The goal of the following analysis is to find how the amplitudes $\vec{m}$ and $\vec{h}$ change in time, by using the Landau-Lifshitz equation. We will find that they oscillate with some frequency ω. This will lead to propagating plane waves of magnetization and magnetic field, as in (18) and (19). The Landau-Lifshitz equation involves the cross product $\vec{M} \times \vec{H}$. From (20) and (21), and some simplification,

$$\vec{M} \times \vec{H} = \left[H_0\hat{z} \times \vec{m}e^{i(\vec{k}\cdot\vec{r})} + \vec{h} \times M_0\hat{z}e^{i(\vec{k}\cdot\vec{r})} + \vec{m} \times \vec{h}e^{2i(\vec{k}\cdot\vec{r})}\right] \tag{22}$$

We are considering spin waves in a linear regime, where the amplitude is small. So, $h \ll H_0$ and $m \ll M_0$. So, the term $\vec{m}\times\vec{h}$ can be ignored because it is very small. From $d\vec{M}/dt = -|\gamma|(\vec{M}\times\vec{H})$, and noting that $M_0\hat{z}$ is constant we have

$$\dot{\vec{m}} = -|\gamma|(H_0\hat{z} \times \vec{m} + \vec{h} \times M_0\hat{z}). \tag{23}$$

The complex exponential factors on both sides of the equation were identical, and canceled. Since $\vec{h}$ is and $\vec{m}$ are plane waves, (17) holds, so we have

$$\dot{\vec{m}} = -|\gamma|\left(H_0(\hat{z} \times \vec{m}) - M_0(\vec{k} \times \hat{z})\frac{(\vec{m}\cdot\vec{k})}{k^2}\right). \tag{24}$$

This is a linear system of differential equations that describes how the vector $\vec{m}$ evolves in time. This system of equations can be represented as

$$\dot{\vec{m}} = \Omega\vec{m}, \tag{25}$$

where the matrix Ω is

$$\Omega = \begin{pmatrix} 0 & (\omega_H + \omega_M sin^2\theta) & \frac{\omega_M}{2}sin(2\theta) \\ -\omega_H & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{26}$$

Here, $\omega_M = |\gamma|M_0$ and $\omega_H = |\gamma|H_0$. These constants have units of frequency. The constant $sin^2\theta$ appears in the form k_y^2/k^2, and it comes from the second term in (24). θ is the angle between the wavevector $\vec{k}$ and the applied field $\vec{H}$. The constant $sin(2\theta)/2$ appears in the form k_yk_z/k^2, and it also comes from the second term in (24). Now that the differential equation that determines $\vec{m}$ has been written in matrix form, it is easy to proceed.

The next step is to find the eigenvalues and eigenvectors of Ω. The eigenvalues are are $\pm i\sqrt{\omega_H(\omega_H + \omega_M sin^2\theta)}$ and 0. From now, $\pm\sqrt{\omega_H(\omega_H + \omega_M sin^2\theta)}$ will be denoted as $\pm\omega$. For the eigenvalue $-i\omega$, the corresponding eigenvector is

$$\vec{V} = \begin{pmatrix} i\tilde{\omega} \\ 1 \\ 0 \end{pmatrix}, \tag{27}$$

where $\tilde{\omega} = \omega/\omega_H$. For the eigenvalue $i\omega$, the corresponding eigenvector is

$$\vec{V}^* = \begin{pmatrix} -i\tilde{\omega} \\ 1 \\ 0 \end{pmatrix}. \tag{28}$$

Finally, for the eigenvalue 0, the corresponding eigenvector is something which has no clear physical meaning. It has nothing to do with an oscillation, since the eigenvalue is 0. Instead, it describes a stable plane wave distribution of magnetization, which does not make sense. Since we have found the frequency of oscillation of a plane wave with a wave vector $\vec{k}$, we have found a dispersion relation for spin waves in an unbounded media. That is $\omega^2 = \omega_H(\omega_H + \omega_M sin^2\theta)$. Note that spin waves are possible at any frequency between $\sqrt{\omega_H(\omega_H + \omega_M)}$ and ω_H. Also note that the spin wave frequency only depends on the direction that the waves are propagating, not on the magnitude of the wavevector. In a finite sample this is not the case, as the modes will be quantized due to the boundary conditions of the sample. The unbounded media is a reasonable approximation if the sample is much larger than the wavelengths of any of the waves under consideration.

This section only offers a brief introduction to the linear theory of spin waves. An interested reader may wish to consult [2] and [3] for a much more complete description.

5 Physical description of linear modes

The goal of this section is to make the physical meaning of the solutions found in the last section more obvious, by writing a typical plane wave solution in a more transparent form. Assume that somehow there was an initial magnetization distribution of the form

$$\vec{m} = m_0 \cos(\vec{k} \cdot \vec{r})\hat{y} \tag{29}$$

This can be written as

$$\vec{m}(\vec{r}, 0) = \frac{m_0}{2}\left[\vec{V} e^{i(\vec{k}\cdot\vec{r})} + \vec{V}^* e^{-i(\vec{k}\cdot\vec{r})}\right]. \tag{30}$$

Note that none of the entries in the matrix Ω change if the vector $\vec{k}$ is replaced by the vector $-\vec{k}$. Had we started the last section assuming a magnetization distribution of the form $\vec{m}(t)e^{-i(\vec{k}\cdot\vec{r})}$, we would have still found a set of linear differential equations for $\vec{m}$, and $\vec{V}^*$ would have still been an eigenvector with an eigenvalue $i\omega$. Since the coefficients of $e^{-i(\vec{k}\cdot\vec{r})}$ evolve with the same differential equations as the coefficients of $e^{i(\vec{k}\cdot\vec{r})}$, an initial magnetization distribution of the form (30) will evolve as

$$\vec{m}(\vec{r}, t) = \frac{m_0}{2}\left[\vec{V} e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \vec{V}^* e^{-i(\vec{k}\cdot\vec{r}+\omega t)}\right]. \tag{31}$$

More explicitly, this can be written as

$$\vec{m}(\vec{r}, t) = \frac{m_0}{2}\left[\begin{pmatrix} i\tilde{\omega} \\ 1 \\ 0 \end{pmatrix} e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \begin{pmatrix} -i\tilde{\omega} \\ 1 \\ 0 \end{pmatrix} e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\right]. \tag{32}$$

Using Euler's equation, this can be written as

$$\vec{m}(\vec{r}, t) = m_0 \begin{pmatrix} -\tilde{\omega}\sin(\vec{k}\cdot\vec{r}-\omega t) \\ \cos(\vec{k}\cdot\vec{r}-\omega t) \\ 0 \end{pmatrix} \tag{33}$$

This form is easy to understand. At any point in space, the magnetization vector $\vec{m}(r,t)$ precesses about the applied field in a clockwise direction. The magnetization moves in an elliptical path with a semiminor axis m_0 and a semimajor axis $\tilde{\omega} m_0$. The phase of this precession changes in space, and increases in the direction of $\vec{k}$. A plane spin wave can be thought of as a propagating wave of magnetization precession. It is perfectly valid to have $\vec{k} = 0$, and this corresponds to a uniform precession throughout the whole sample.

6 Nonlinear amplitude equations

It has been seen in experiments that spin waves undergo three-wave interactions. These interactions involve three modes, and these modes will have wavevectors $\vec{k}_1$,$\vec{k}_2$ and $\vec{k}_3$. This section will use the Landau-Lifshitz equation to explore their interactions. These waves satisfy conservation of momentum:

$$\vec{k}_1 = \vec{k}_2 + \vec{k}_3, \tag{34}$$

and conservation of energy

$$\omega_1 = \omega_2 + \omega_3, \tag{35}$$

Similar to the earlier section, the magnetic field and magnetization vectors will be assumed to have a spatial dependence of the following form:

$$\vec{H} = H_0 \hat{z} + \sum_{q=1}^{3} \left[\vec{h}_q e^{i(\vec{k}_q \cdot \vec{r})} + \vec{h}_q^* e^{-i(\vec{k}_q \cdot \vec{r})} \right] \tag{36}$$

$$\vec{M} = M_0 \hat{z} + \sum_{p=1}^{3} \left[\vec{m}_p e^{i(\vec{k}_p \cdot \vec{r})} + \vec{m}_p^* e^{-i(\vec{k}_p \cdot \vec{r})} \right]. \tag{37}$$

It is important that the magnetic field and the magnetization be explicitly real, and so the complex conjugates are added. Also, the wavevectors $\vec{k}_1$,$\vec{k}_2$ and $\vec{k}_3$ will all be assumed to be in the yz plane. In Section 4, there was no loss of generality from assuming that the wave under consideration had a wavevector in the yz plane. Now, there is a loss of generality, as all of the wavevectors are assumed to lie in the same plane. Once again, we will take the cross product $\vec{M} \times \vec{H}$.

$$\begin{aligned} \vec{M} \times \vec{H} = M_0 \sum_{q=1}^{3} \left[(\hat{z} \times \vec{h}_q) e^{i(\vec{k}_q \cdot \vec{r})} + (\hat{z} \times \vec{h}_q)^* e^{-i(\vec{k}_q \cdot \vec{r})} \right] + H_0 \sum_{p=1}^{3} \left[(\vec{m}_p \times \hat{z}) e^{i(\vec{k}_p \cdot \vec{r})} + (\vec{m}_p^* \times \hat{z}) e^{-i(\vec{k}_p \cdot \vec{r})} \right] + \\ \underbrace{\left\{ \sum_{p=1}^{3} \left[\vec{m}_p e^{i(\vec{k}_p \cdot \vec{r})} + \vec{m}_p^* e^{-i(\vec{k}_p \cdot \vec{r})} \right] \right\} \times \left\{ \sum_{q=1}^{3} \left[\vec{h}_q e^{i(\vec{k}_q \cdot \vec{r})} + \vec{h}_q^* e^{-i(\vec{k}_q \cdot \vec{r})} \right] \right\}}_{\Sigma} \end{aligned} \tag{38}$$

This can be inserted into $d\vec{M}/dt = -|\gamma|(\vec{M} \times \vec{H})$. After doing this, terms with common spatial dependence are gathered together. This will now result in three sets of differential equations. Note that Σ is contains a huge number of terms with complex exponential factors of the form $e^{i(\pm\vec{k}_p \pm \vec{k}_q)\cdot\vec{r}}$. Since the wavevectors $\vec{k}_p$ obey the condition (34), many of the complex exponential factors of the terms in Σ will reduce to a simpler form if the wavevectors involved add to give the wavevector of another mode. For example, if a term has a coefficient of the form $e^{i(\vec{k}_2+\vec{k}_3)\cdot\vec{r}}$, this can be simplified to $e^{i\vec{k}_1\cdot\vec{r}}$. Because of terms like this, plane waves do not necessarily propagate indepentently. Energy can flow between these modes. After expanding $(\vec{M} \times \vec{H})$, and after collecting terms with the same complex exponential factors, there are the following three sets of differential equations that describe the evolution of the three modes:

$$\dot{\vec{m}}_1 = \Omega_1 \vec{m}_1 - |\gamma|[\vec{m}_2 \times \vec{h}_3 + \vec{m}_2 \times \vec{h}_3] \tag{39}$$
$$\dot{\vec{m}}_2 = \Omega_2 \vec{m}_2 - |\gamma|[\vec{m}_1 \times \vec{h}_3^* + \vec{m}_3^* \times \vec{h}_1] \tag{40}$$
$$\dot{\vec{m}}_3 = \Omega_3 \vec{m}_3 - |\gamma|[\vec{m}_1 \times \vec{h}_2^* + \vec{m}_2^* \times \vec{h}_1] \tag{41}$$

The matrix Ω_1 is defined as the matrix Ω was earlier, only with $\vec{k}_1$ in place of $\vec{k}$, and so on. Now, (17) can be used to eliminate the $\vec{h}$'s in these equations.

$$\dot{\vec{m}}_1 = \Omega_1 \vec{m}_1 + |\gamma| \left[\vec{m}_2 \times \vec{k_3} \frac{(\vec{m}_3 \cdot \vec{k}_3)}{k_3^2} + \vec{m}_3 \times \vec{k_2} \frac{(\vec{m}_2 \cdot \vec{k}_2)}{k_2^2} \right] \tag{42}$$

$$\dot{\vec{m}}_2 = \Omega_2 \vec{m}_2 + |\gamma| \left[\vec{m}_1 \times \vec{k_3} \frac{(\vec{m}_3^* \cdot \vec{k}_3)}{k_3^2} + \vec{m}_3^* \times \vec{k_1} \frac{(\vec{m}_1 \cdot \vec{k}_1)}{k_1^2} \right] \tag{43}$$

$$\dot{\vec{m}}_3 = \Omega_2 \vec{m}_3 + |\gamma| \left[\vec{m}_1 \times \vec{k_2} \frac{(\vec{m}_2^* \cdot \vec{k}_2)}{k_2^2} + \vec{m}_2^* \times \vec{k_1} \frac{(\vec{m}_1 \cdot \vec{k}_1)}{k_1^2} \right] \tag{44}$$

It is possible that these equations can be written in a more illuminating form if one assumes a that the magnetization amplitudes $\vec{m}$ have the form:

$$\vec{m} = a\vec{V} + b\hat{z}. \tag{45}$$

Here, a and b are complex numbers. This is a very general form, as any real vector can be written as $\vec{u} = (a\vec{V} + b\hat{z}) + (a\vec{V} + b\hat{z})_*$. Writing the magnetization amplitude in this form, (42-44) become:

$$a_1 = -i\omega_1 a_1 + A_1 b_1 + |\gamma|(I_1 + J_1) \tag{46}$$
$$b_1 = |\gamma|(K_1 + L_1) \tag{47}$$
$$a_2 = -i\omega_2 a_2 + A_2 b_2 + |\gamma|(I_2 + J_2) \tag{48}$$
$$b_2 = |\gamma|(K_2 + L_2) \tag{49}$$
$$a_3 = -i\omega_3 a_3 + A_3 b_3 + |\gamma|(I_3 + J_3) \tag{50}$$
$$b_3 = |\gamma|(K_3 + L_3) \tag{51}$$

with

$$A_p = \left[\frac{-i\tilde{\omega}_p}{1 + \tilde{\omega}_p^2} \right] \frac{\omega_M}{2} sin(2\theta_p) \tag{52}$$

These equations came from inserting (45) into (42-44), and projecting the resulting equations onto the vectors $\vec{V}$ and $\hat{z}$. The $A_p b_p$ terms come from the fact that $\hat{z}$ is not a eigenvector of Ω. When the $b\hat{z}$ term in (45) is acted on my the matrix Ω, the result is a term which, when projected along $\vec{V}$ gives $A_p b_p$. The I's, J's are projections of nonlinear terms in (42-44) onto $\vec{V}$, and K's and L's are projections of nonlinear terms in (42-44) onto $\hat{z}$. For mode 1, the nonlinear terms are defined as

$$I_1 = \frac{\left\langle \vec{V}, \vec{m}_2 \times \vec{k_3} \frac{(\vec{m}_3 \cdot \vec{k}_3)}{k_3^2} \right\rangle}{|\vec{V}|^2} \tag{53}$$

$$J_1 = \frac{\left\langle \vec{V}, \vec{m}_3 \times \vec{k_2} \frac{(\vec{m}_2 \cdot \vec{k}_2)}{k_2^2} \right\rangle}{|\vec{V}|^2} \tag{54}$$

$$K_1 = \left\langle \hat{z}, \vec{m}_2 \times \vec{k_3} \frac{(\vec{m}_3 \cdot \vec{k}_3)}{k_3^2} \right\rangle \tag{55}$$

$$L_1 = \left\langle \hat{z}, \vec{m}_3 \times \vec{k_2} \frac{(\vec{m}_2 \cdot \vec{k}_2)}{k_2^2} \right\rangle \tag{56}$$

The inner product used here is the standard complex inner product $\langle \vec{a}, \vec{b} \rangle = \sum \vec{a}_i^* \vec{b}_i$, and the other terms are defined in a completely analogous way. With some calculation, one finds:

$$I_1 = \frac{-i(a_3 k_{y3} + b_3 k_{z3})}{k_3^2 (1 + \tilde{\omega}_1^2)} \left[k_{z3}(\tilde{\omega}_1 + \tilde{\omega}_2 a_2 + \tilde{\omega}_1 k_{y3} b_2) \right] \tag{57}$$

$$K_1 = \frac{i\tilde{\omega} k_{z3}(a_3 k_{y3} + b_3 k_{z3})}{k_3^2} \tag{58}$$

It would be easy to obtain the other nonlinear terms from these by changing the subscripts involved and replacing $\tilde{\omega}$'s by $-\tilde{\omega}$'s when the $\tilde{\omega}$ term comes from a vector that shows up in (42-44) as a complex conjugate. Clearly, it is possible to create nonlinear amplitude equations which model the three wave interactions, but they are obviously very complicated, and perhaps not useful.

7 Closing remarks

The nonlinear amplitude equations (46-51) describe the three-wave interactions of plane waves propagating in an unbounded ferromagnetic sample. It would be interesting to numerically integrate these equations to see if they at least qualitatively reproduce any of the results in [1]. It would also be possible to phenomenolically include the damping-driving dynamics of the real system into these equations by adding an imaginary part to the frequency ω. This is done in [2]. It is also possible that the equations (46-51) are much too complicated to study in a useful way, and some simplifying assumtions must be made.

References

[1] A. M. Hagerstrom, W. Tong, M. Wu, B. A. Kalinikos, and R. Eykholt , "Excitation of chaotic spin waves in magnetic film feedback rings through three-wave nonlinear interactions," Phys. Rev. Lett. 102, 207202 (2009).

[2] Magnetic Oscillations and Waves, A.G. Gurevich G.A Melkov

[3] Theory of Magnetostatic Waves, Daniel D. Stancil