Modeling Soliton Solutions to the Sine-Gordon Equation

Jeff Lyons, Heith Kippenhan, Erik Wildforster
Department of Physics
Colorado State University

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Abstract. We exploit the nonlinear effects of rubberbands in a physical system consisting of pendula attached to a horizontal rod. We discuss pseudospherical surfaces and their relation to the sine-Gordon equation. We further discuss soliton solutions to the sine-Gordon equation as well as the Bäcklund Transformation and how our pendula system is used to demonstrate soliton solutions to the sine-Gordon. We also mention applications of this system to physics, including Lorentz invariance.

Keywords: sine-Gordon, solitons, Backlund transform

1 Introduction

Differential geometry is a rich area of mathematics that has applications in many fields. In this paper we explore the application of various principles from differential geometry to pseudospherical surfaces, obtaining the sine-Gordon equation. We then investigate soliton solutions of the sine-Gordon equation by means of the Backlund transformation and describe a system of pendula built to demonstrate specific solutions to this equation. We then explore a few of the implications of a universe governed by the sine-Gordon equation and show that because it is Lorentz-invariant, the effects of the special theory of relativity are apparent and demonstrable in our pendula system.

2 Pseudospherical Surfaces

The classic pseudosphere can be understood by considering a curve, known as a tractrix, parameterized by \( t \rightarrow (t - \tanh t, \sech t) \), \( 0 \leq t < \infty \). Revolving this curve about its asymptote generates a hyperbolic surface, such as that shown in Fig. 1, having many similarities to the traditional sphere. The sphere has a constant Gaussian curvature related to its radius by \( K = \frac{1}{R^2} \), has an area \( A = 4\pi R^2 \), and a volume of \( V = \frac{4}{3}\pi R^3 \). The characteristic feature of the pseudosphere is that it has constant negative Gaussian curvature and can be ascribed some sort of "radius" through the relation \( K = -\frac{1}{\rho^2} \). It has been shown that such a surface has a surface area \( A = 4\pi \rho^2 \) and a volume \( V = \frac{2}{3}\pi \rho^3 \).
3 Sine-Gordon Equation

We can alternatively describe the pseudosphere described in the last section by means of a map \( \vec{x}(u, v) \) from a patch to the surface. If this map is parameterized by arclength along asymptotic lines then the first fundamental form for the pseudosphere has the form \( I = d\vec{x} \cdot d\vec{x} = du^2 + 2\cos \phi du dv + dv^2 \). Similarly, the second fundamental form is \( II = -d\vec{x} \cdot d\vec{N} = \frac{2}{\rho} \sin \phi du dv \). Application of the Codazzi-Mainardi equations then yields \( \phi_{uv} = \frac{1}{\rho^2} \sin \phi \), the celebrated sine-Gordon equation. Pseudospherical surfaces then, can be thought of as solutions to the sine-Gordon equation.

4 Soliton Solutions and the Bäcklund Transformation

Bäcklund transformations relate partial differential equations and their solutions. Of particular interest are auto-Bäcklund transforms, which relate solutions of the same equation. They were first introduced by L. Bianchi and A.V. Bäcklund in the 1880s as a way to find other pseudospherical surfaces from an initial pseudosphere and a solution of a linear differential equation. Because pseudospherical surfaces are solutions of the sine-Gordon equation, the Bäcklund transform can be viewed as transforming solutions to the sine-Gordon equation. In particular, if \( u \) is a solution of the sine-Gordon equation, \( u_{xy} = \sin u \), then the auto-Bäcklund transformation is the system

\[
\begin{align*}
    v_x &= u_x + 2a \sin \left( \frac{u + v}{2} \right) \\
    v_y &= -u_y + \frac{2}{a} \sin \left( \frac{v - u}{2} \right)
\end{align*}
\]

where \( a \) is an arbitrary parameter. This system can be used to generate a solution \( v \) that satisfies the sine-Gordon from the solution \( u \). Application of this transform has been found to be particularly useful in soliton theory since it can be used to generate an infinite number of soliton solutions if one is known. Such is the case with the sine-Gordon equation. Since the trivial solution \( (\phi = 0) \) is a (soliton) solution to the sine-Gordon, it can then be used to generate other soliton solutions. Using \( u = 0 \) in the system above yields 1-soliton solutions of the form \( \phi(x, t) = 4 \arctan(e^{\gamma(x-\nu t)+\delta}) \), expressed in space-time coordinates, where \( \gamma^2 = \frac{1}{\nu^2} \). Taking the positive root for \( \gamma \) yields 1-soliton solutions known as kinks, which takes the variable \( \phi \) to states \( \phi = 0 (\mod 2\pi) \). The negative root is known as an anti-kink. Through continued application of the Bäcklund transform to 1-soliton solutions, 2-soliton solutions can be obtained, one of the more interesting being the breather.
5 Pendula System

Our project attempts to demonstrate 1-soliton solutions of the sine-Gordon equation via the elastic ribbon model. Our setup, shown in Fig. 2, consists of a wooden stand about two feet in width at the legs, two and a half feet in length, and stands about a foot off the ground. We drilled holes in the stand and placed bearings to support a 5/8" diameter steel rod on which rest the pendula. The pendula were made from bolts and nuts welded together and milled out to allow the pendula to swing freely. We used extra nuts as spacers between the pendula and connected the pendula with rubber bands, held together with nuts fitting the bolts. The steel rod was lathed down to fit the drill, which was used as a motor to drive the system. Due to the fact that rubberbands exhibit a nonlinear restorative force of the form $F = -kx + ax^2 + bx^3 + \ldots$, this system is able to demonstrate the nonlinear properties of soliton solutions to the sine-Gordon equation.

6 Lorentz Invariance and Physics Applications

One neat thing that arises from this setup is the demonstration of special relativistic effects such as time dilation and length contraction from soliton solutions of the sine-Gordon. The reason this is demonstrable is that the sine-Gordon equation is invariant under Lorentz transformations. The Lorentz transformations are

$$t' = \gamma(t - \frac{vx}{c^2})$$
$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$

and the inverse Lorentz transformations obtained simply by interchanging the primed variables with the unprimed and adding instead of subtracting. The sine-Gordon equation can be expressed in space-time coordinates as

$$\phi_{tt} - \phi_{xx} + \sin \phi = 0$$
We can re-express this in the primed coordinates by recognizing that \( \phi_t' = \phi_t \frac{\partial}{\partial t'} = \gamma \phi_t \), 
\[ \phi_{tt'} = \frac{\partial^2}{\partial t'^2} \phi_t + \gamma \phi_t \frac{\partial}{\partial t'} = \gamma^2 \phi_{tt} \] since \( \gamma \) is constant. And similarly for the space-coordinate, 
\[ \phi_{xx'} = \gamma^2 \phi_{xx}. \] Thus the transformed sine-Gordon is

\[ \phi_{tt'} - \phi_{xx'} + \frac{1}{\gamma^2} \sin \phi = 0 \]

We can assume that the basic laws of the universe are governed by the sine-Gordon equation. In such a universe we would measure objects spatially by means of kink solutions to the sine-Gordon equation. As the velocity increases, the kinks shorten, demonstrating the relativistic effect of length contraction. To measure time, breather solutions of the sine-Gordon provide a periodic motion which can be used to give a reference for simultaneous events. As velocity is increased, the period increases, which is the relativistic effect of time dilation.
References

