MATH 670 Intro to Manifolds : Suggested Project Topics

Below are some suggested topics for projects. Some of these are quite difficult and I certainly don't expect you to understand how they are proved; in some cases, it would be enough to understand the statement of the theorem, which would already require understanding some important concepts. Please feel free to discuss these with me and ask any questions you might have. I can suggest where you might find more details in the literature (Wikipedia and Mathworld are good starting places, but you should venture beyond these).

1. **Topological manifolds which don't admit smooth structures:** Kervaire (Comm. Math. Helv. 34 (1960), 304–312) and Smale (Annals of Math. 74 (1961), 391–406) found examples of topological manifolds which don't admit any smooth structures. Find out how these examples are constructed and why they don't admit smooth structures. Do such examples exist in any dimension?

2. **Exotic smooth structures on seven-dimensional spheres:** Milnor (Annals of Math. 64 (1956), 399–405) found 7-dimensional manifolds which are homeomorphic to the sphere S^7 but not diffeomorphic to S^7. Find out how these examples are constructed. How many exotic seven-spheres are there?

3. **Classification of topological four-manifolds:** Let X be a compact simply-connected four-manifold. Freedman (J. Diff. Geom. 17 (1982), no. 3, 357–453) proved that the topology of X is essentially determined by the intersection form on the middle dimensional homology, and any intersection form can occur. Find out what this statement means and how it relates to the four-dimensional Poincaré conjecture.

4. **Donaldson theory in four dimensions:** In contrast to the previous topic, Donaldson (J. Diff. Geom. 18 (1983), no. 2, 279–315) showed that there are restrictions on the intersection form of a smooth manifold. Find out what these restrictions are and how Donaldson found them. Donaldson also showed that there are exotic smooth structures on $\mathbb{R}^4$; how does one construct these and show they are different to the standard smooth structure?

5. **Manifolds of intermediate differentiability C^1, C^2, etc.:** For $1 \leq r \leq s \leq \infty$, any C^r manifold is C^r diffeomorphic to a C^s manifold, and the latter manifold is unique up to a C^s diffeomorphism; so there is no great difference between C^r and C^s manifolds. This is in contrast to the previous topics, which illustrate that C^0 (topological) manifolds can be very different to C^∞ (smooth) manifolds. Find out how these smoothability results work.

6. **Complex manifolds and almost complex manifolds:** A complex manifold has local coordinate charts which map onto open subsets of $\mathbb{C}^n$, and changes of coordinates should be holomorphic maps. The tangent space at any point then becomes a complex vector space, so there is a linear operator I (multiplication by $\sqrt{-1}$) on each fibre. An almost complex manifold is a smooth manifold X with a linear operator $I : T_pX \to T_pX$ on each tangent space, such that $I^2 = -\text{identity}$. Thus any complex manifold admits an almost complex structure. Conversely, the Newlander-Nirenberg Theorem says that an almost

complex manifold is a complex manifold if and only if the almost complex structure is *integrable*. Find out what this means and why integrability is needed.

7. **Gromov-Hausdorff distance between topological spaces:** We know what it means for two topological spaces X and Y to be "the same" (they are homeomorphic), but how could we define "almost the same"? If X and Y are subspaces of the some larger metric space Z, then we could say X and Y are distance $\leq \epsilon$ from each other if every point of X is no further than distance ϵ from a point of Y, and vice versa. For general topological spaces X and Y, we first need to embed them as subspaces of a metric space Z. Taking a limit over all possible embeddings (and all possible Z) leads to the notion of Gromov-Hausdorff distance. Find out the precise the definition and some of its applications.

8. **A C^1 counterexample to Sard's theorem:** Whitney (Duke Math. J. 1 (1935), 514–517) found an example of a C^1 function on $\mathbb{R}^2$ whose critical values in $\mathbb{R}$ contain an interval, and therefore have positive measure. The proof involves some kind of recursion leading to fractal-like behaviour. Find out how this function is constructed.

9. **Non-embeddings results:** The Whitney Embedding Theorem tells us that the Klein bottle can be embedded in $\mathbb{R}^4$. It can't be embedded in $\mathbb{R}^3$, though one generally needs methods from algebraic topology to show this kind of result. Suppose that the n-dimensional manifold M can be embedded in $\mathbb{R}^N$ for some N. The idea is to look at the short exact sequence

$$0 \to TM \to T\mathbb{R}^N|_M \to N_{M\subset\mathbb{R}^N} \to 0.$$

One now takes characteristic classes (e.g., Stiefel-Whitney classes) to get

$$w(TM)w(N_{M\subset\mathbb{R}^N}) = w(T\mathbb{R}^N|_M) = 1$$

since $T\mathbb{R}^N|_M \cong M \times \mathbb{R}^N$ is trivial. If we know $w(TM)$, then we can sometimes reach a contradiction if the rank $N - n$ of the normal bundle $N_{M\subset\mathbb{R}^N}$ is too small. Find some examples where this method can be applied (e.g., can you show that $\mathbb{CP}^2$ can't be embedded in $\mathbb{R}^5$?).