MATH 670 Intro to Manifolds : Exercise Sheet Four

1. Let $\mathbb{R} + S^1$ be the disconnected manifold consisting of the disjoint union of the real line and a circle. Define $f : \mathbb{R} + S^1 \to \mathbb{C} \cong \mathbb{R}^2$ by

$$\begin{aligned} f(x) &= (1 + \exp(x)) \exp(ix) \qquad \text{for} \qquad x \in \mathbb{R}, \\ f(\theta) &= \exp(i\theta) \qquad\qquad\qquad\quad \text{for} \qquad \theta \in S^1. \end{aligned}$$

   Prove that f is an immersion which is globally injective, but not an embedding.

2. Define the map $f : \mathbb{R} \to \mathbb{C} \times \mathbb{C} \cong \mathbb{R}^4$ by

$$x \mapsto (\exp(aix), \exp(bix))$$

   for real numbers a and b. Show that f is an immersion provided a or b is nonzero. Assume $b \neq 0$ and a/b is irrational; show that f is globally injective. Is f an embedding?

3. Define the smooth map

$$f : \mathbb{R}^{n^2} \to S$$

   from the space of $n \times n$-matrices to the space of symmetric matrices by $f(A) = {}^tAA$. Complete the proof that the *orthogonal group* $\mathrm{O}(n)$ of real orthogonal $n \times n$-matrices is a smooth submanifold of $\mathbb{R}^{n^2}$ by showing that the identity matrix is a regular value of f.

4. Let $0 < k < n$. A k-frame in $\mathbb{R}^n$ is an *orthonormal* k-tuple $(v_1, \ldots, v_k)$ of vectors in $\mathbb{R}^n$. The set V_n^k of k-frames in $\mathbb{R}^n$ is called a *Stiefel manifold*. Show that V_n^k is a compact smooth manifold of dimension $nk - k(k+1)/2$.

5. [Extra Credit] i) Let Δ be a small disc on the torus $S^1 \times S^1$. Show that there is an immersion of the punctured torus $S^1 \times S^1 \backslash \Delta$ into $\mathbb{R}^2$.

   ii) Is there an embedding of the punctured torus in $\mathbb{R}^2$?