MATH 670 Intro to Manifolds : Exercise Sheet Two

1. (i) By using the coordinate charts coming from stereographic projection, show that the tangent bundle of S^2 possesses an atlas with two bundle charts.

   (ii) Construct a vector field on S^2 with exactly two zeroes, at the North and South poles. What does this vector field look like in the two bundle charts?

   (iii) Construct a vector field on S^2 with exactly one zero at the North pole. What does this vector field look like in the two bundle charts?

2. Prove that every rank one bundle on S^1 is either trivial ($E \cong S^1 \times \mathbb{R}$) or isomorphic to the Möbius band (a cylinder with a single twist).

   [Notice that a cylinder with two twists will be isomorphic to the trivial bundle.]

3. Let
$$E := \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} | z_0^2 + \ldots + z_n^2 = 1\}.$$
   By writing $z_k = x_k + iy_k$ in terms of real and imaginary parts we can regard E as a submanifold of $\mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$, i.e.,

   $$E = \{(x_0, y_0, \ldots, x_n, y_n) \in \mathbb{R}^{2n+2} | x_0^2 - y_0^2 + \ldots + x_n^2 - y_n^2 = 1 \quad \text{and} \quad x_0 y_0 + \ldots + x_n y_n = 0\}.$$

   Find an explicit diffeomorphism from E to the total space of the tangent bundle of S^n.

4. Let E_1 and E_2 be the trivial rank one and rank two bundles over $\mathbb{R}^2$, respectively. Define a smooth map $f : E_2 \to E_1$ by
$$\begin{array}{rccc} f_{(x,y)} & : & (E_2)_{(x,y)} & \to & (E_1)_{(x,y)} \\ & & (x, y, \mu, \nu) & \mapsto & (x, y, x\mu + y\nu) \end{array}$$
   where $(x, y) \in \mathbb{R}^2$. Describe kernel$f$. Is it a vector bundle in a neighbourhood of the origin $(0, 0) \in \mathbb{R}^2$?