MATH 670 Intro to Manifolds : Exercise Sheet Ten

1. Let M^m and N^n be compact oriented submanifolds of $\mathbb{R}^{k+1}$ which are also disjoint. Define the *linking map*

$$\begin{aligned} \lambda : M \times N &\to S^k \\ (x, y) &\mapsto \frac{x-y}{\|x-y\|}. \end{aligned}$$

If $m + n = k$, i.e., $\dim(M \times N) = \dim S^k$, then the *linking number* $l(M, N)$ of M and N is defined to be the degree of λ.

(i) Prove that

$$l(M, N) = (-1)^{(m+1)(n+1)} l(N, M).$$

(ii) Suppose there exists an oriented submanifold $X \subset \mathbb{R}^{k+1}$ with boundary $\partial X = M$ and such that X does not intersect N. Prove that $l(M, N) = 0$.

2. Let y and z be distinct regular values of a smooth map $f : S^{2p-1} \to S^p$. Then $f^{-1}(y)$ and $f^{-1}(z)$ are smooth oriented submanifolds of S^{2p-1}. The linking number of Exercise 1 can be defined for submanifolds of the sphere by stereographically projecting from a point which is not on either submanifold; hence the linking number $l(f^{-1}(y), f^{-1}(z))$ is well-defined.

One can show that $l(f^{-1}(y), f^{-1}(z))$ depends only on the homotopy class of the map f and is independent of the choice of regular values y and z. The integer

$$H(f) := l(f^{-1}(y), f^{-1}(z))$$

is called the *Hopf invariant* of f.

(i) If p is odd, prove that $H(f)$ must vanish.

(ii) Recall the Hopf fibration

$$\pi : S^3 \to S^2$$

which is given by mapping

$$(z_1, z_2) \in S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$$

to

$$[z_1, z_2] \in \mathbb{CP}^1 \cong S^2.$$

Prove that $H(\pi) = 1$.