MATH 670 Intro to Manifolds : Exercise Sheet One

1. In "Topology from a differentiable viewpoint" Milnor starts with a subset $M \subset \mathbb{R}^k$ when defining a manifold. In this case, $\mathbb{R}^k$ has the usual topology coming from the Euclidean metric and M inherits the subspace topology. Explain why M is Hausdorff, paracompact, and second countable.

   [Later we will see that any smooth manifold M can be embedded in $\mathbb{R}^k$ for some k (Whitney's Theorem) and therefore there is no loss of generality in the above approach.]

2. Using stereographic projection, show that the sphere S^2 can be covered by two coordinate charts (U_1, h_1) and (U_2, h_2) where U_1 is the complement of the North Pole and U_2 is the complement of the South Pole. Calculate the coordinate transformation between these two charts, and hence show that they give a smooth atlas.

3. (i) Show that the Cartesian product $f_1 \times f_2 : N_1 \times N_2 \to M_1 \times M_2$ of two embeddings is an embedding.

   (ii) Let the n-dimensional manifold M be a product of spheres

   $$S^{k_1} \times S^{k_2} \times \cdots \times S^{k_t},$$

   where $n = k_1 + k_2 + \ldots + k_t$. Show that M can be embedded in $\mathbb{R}^{n+1}$. For example, the torus $S^1 \times S^1$ can be embedded in $\mathbb{R}^3$.

   [Hint: First describe an embedding $S^n \times \mathbb{R} \to \mathbb{R}^{n+1}$, then use part (i).]

4. Let $f : N \to M$ be an embedding with $f(p) = q$. Show that $f^* : \mathcal{E}(q) \to \mathcal{E}(p)$ is surjective and $T_p f$ is injective.

5. [Extra credit] Let $M \subset \mathbb{R}^2$ be the square with vertices $(\pm 1, \pm 1)$. Is M a smooth manifold? (More precisely, can M be covered by a smooth atlas?)