

MATH567 Abstract Algebra II : Solutions to Sec. 10.4, exercises 3–4.

3. Firstly, \mathbb{C} is both a left \mathbb{R} -module and left \mathbb{C} -module, simply by multiplication on the left, and also a right \mathbb{R} -module and right \mathbb{C} -module, by multiplication on the right. Regarding the first \mathbb{C} as a (\mathbb{R}, \mathbb{R}) -bimodule and the second \mathbb{C} as a left \mathbb{R} -module, we see that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is a left \mathbb{R} -module. Similarly, regarding the first \mathbb{C} as a (\mathbb{R}, \mathbb{C}) -bimodule and the second \mathbb{C} as a left \mathbb{C} -module, we see that $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ is a left \mathbb{R} -module.

The \mathbb{R} -module $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is discussed in the example on page 375. In particular, it is a free module of rank 4 over \mathbb{R} with basis given by $1 \otimes 1$, $1 \otimes i$, $i \otimes 1$, and $i \otimes i$. By contrast, $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ will be a free module of rank 2 over \mathbb{R} : $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ is isomorphic to \mathbb{C} as a left \mathbb{C} -module (in general, $R \otimes_R M \cong M$ for any left R -module M). Restricting the action to $\mathbb{R} \subset \mathbb{C}$, we see that \mathbb{C} is a free module of rank 2 over \mathbb{R} . In particular, we can write any element as

$$(a+bi) \otimes (c+di) = (a+bi)(c+di) \otimes 1 = ((ac-bd) + (ad+bc)i) \otimes 1 = (ac-bd)1 \otimes 1 + (ad+bc)i \otimes 1$$

where a , b , c , and d are real numbers; so $1 \otimes 1$ and $i \otimes 1$ actually give a basis for $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ as a free module over \mathbb{R} . Note that $1 \otimes i = i \otimes 1$ and $i \otimes i = -1 \otimes 1$.

4. Since $R \otimes_R M \cong M$ for any left R -module M , we immediately have that $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \cong \mathbb{Q}$ is a free module of rank one over \mathbb{Q} . Similarly, a typical element in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ can be written

$$\frac{a}{b} \otimes \frac{c}{d} = \frac{ac}{b} \otimes \frac{1}{d} = \frac{acd}{bd} \otimes \frac{1}{d} = \frac{ac}{bd} \otimes \frac{d}{d} = \frac{ac}{bd} \otimes 1 = \frac{ac}{bd} 1 \otimes 1.$$

It remains to show that $1 \otimes 1$ is non-trivial.

Define a map $\phi : \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{Q}$ by

$$\left(\frac{a}{b}, \frac{c}{d} \right) \mapsto \frac{ac}{bd}.$$

It is easily checked that ϕ is \mathbb{Z} -bilinear, and therefore factors through a map from the tensor product $\Phi : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow \mathbb{Q}$. Since ϕ is surjective, Φ must also be surjective. In particular, $1 \otimes 1$ is mapped to $1 \in \mathbb{Q}$ so $1 \otimes 1$ is non-trivial. Moreover, Φ is an isomorphism (a priori as a map between abelian groups, but the left \mathbb{Q} -actions also agree, so Φ is an isomorphism of left \mathbb{Q} -modules). Therefore

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q} \cong \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}.$$