MAT 569 Differential Geometry II : Exercise Sheet Two

1. Show that a Riemannian manifold with parallel Ricci tensor has constant scalar curvature. Is the converse true? (Prove it or find a counter-example.)

2. Suppose we have two Riemannian manifolds M and N, and put the product metric on $M \times N$. Show that there exist points $p \in M \times N$ and planes $\sigma \in T_p(M \times N)$ such that the sectional curvature $K_p(\sigma)$ vanishes. (Of course, we can put other metrics on $M \times N$, which may not have this property.)

3. Calculate the sectional curvature, Ricci curvature, and the scalar curvature of the product $S^n(a) \times S^m(b)$ of two spheres of dimensions n and m and radii a and b respectively. [Hint: Choose orthonormal frames $E_1, \ldots, E_n$ and $E_{n+1}, \ldots, E_{n+m}$ for each sphere. Then use the result of the previous question, and the fact that spheres have constant curvature.]

4. A Riemannian manifold is *homogeneous* if the isometry group acts transitively. Show that homogeneous manifolds are geodesically complete.

5. Show that any Riemannian manifold admits a complete Riemannian metric.

6. Let N be a submanifold of a Riemannian manifold M. The distance from N to a point x in M is defined as
$$d(x, N) := \inf\{d(x, p) | p \in N\}.$$

   (i) A curve $\sigma : [a, b] \to M$ parametrized by arc-length, and with $\sigma(a) \in N$, $\sigma(b) = x$, and $\ell(\sigma) = d(x, N)$ is called a *segment* from x to N. Show that σ is also a segment from $\sigma(t)$ to N, for all $t < b$. Show that σ is perpendicular to N.

   (ii) If N is closed and M is complete, show that any point x in M can be joined to N by a segment.

   (iii) In general, show that there is an open neighbourhood of N in M where all points can be joined to N by segments (c.f. tubular neighbourhoods).

   (iv) Show that the function $f : x \mapsto d(x, N)$ is smooth in a neighbourhood of N and the integral curves for the gradient vector field ∇f are geodesics perpendicular to N.

7. Let $p \in M$, v and $w \in T_pM$. Show that the distance between the points $\exp_p(tv)$ and $\exp_p(tw) \in M$ is
$$|t(v - w)| + O(t^2).$$

8. Show that if $f : M \to N$ is an isometry, then
$$d(f(p), f(q)) = d(p, q)$$
for all p and $q \in M$.

9. Let $f : (-\epsilon, \epsilon) \times [0, a] \to M$ be a variation of $c : [0, a] \to M$ such that for all $t_0 \in [0, a]$ the transversal curve $s \mapsto f(s, t_0)$ is a geodesic which is parametrized by arc length and orthogonal to the curve c at $f(0, t_0)$. Prove that for all $(s_0, t_0) \in (-\epsilon, \epsilon) \times [0, a]$ the curves $s \mapsto f(s, t_0)$ and $t \mapsto f(s_0, t)$ are orthogonal.

10. If M has zero sectional curvature, show that $\exp_p : B_\epsilon(0) \subset T_pM \to B_\epsilon(p)$ is an isometry for all $p \in M$, where $B_\epsilon(p)$ is a normal ball at p.

11. If M has non-positive sectional curvature, show that there are no pairs of conjugate points.

12. Let M be a Riemannian surface, $p \in M$, and $V \subset T_pM$ be a neighbourhood of the origin such that $\exp_p|_V$ is a diffeomorphism. Let $S_r(0) \subset V$ be a circle of radius r and let L_r be the length of $\exp_p(S_r(0))$ in M. Prove that the sectional curvature at $p \in M$ is given by
$$K(p) = \lim_{r\to 0} \frac{3}{\pi}\frac{2\pi r - L_r}{r^3}.$$
[Hint: See Exercises 6 and 7 in do Carmo's book, page 122.]

13. Let $\gamma : [0, a] \to M$ be a geodesic and let X be a Killing field on M (c.f. Exercise 7 on the first problem sheet).

    (i) Show that X restricted to γ is a Jacobi field.

    (ii) Show that if M is connected and there exists $p \in M$ such that $X(p) = 0$ and $\nabla_Y X(p) = 0$ for all $Y_p \in T_pM$, then $X \equiv 0$ on M (c.f. 7 part (iv) on the first problem sheet).

14. Let $f : M_1 \to M_2$ be a local diffeomorphism of a manifold M_1 onto a Riemannian manifold M_2. Introduce on M_1 a metric such that f is a local isometry. Find an example where M_2 is complete, but M_1 is not complete.

15. (i) Show that the upper half-plane model $\mathbb{R}^2_+$ of two-dimensional hyperbolic space, with the metric
$$ds^2 = \frac{dx^2 + dy^2}{y^2},$$
is complete.

    (ii) Show that $\mathbb{R}^2_+$ with the metric
$$ds^2 = dx^2 + \frac{dy^2}{y}$$
is not complete. [Hint: Calculate the length of the vertical segment from $(0, \epsilon)$ to $(0, 1)$, for $\epsilon > 0$.]

16. A *divergent curve* is a smooth map $\alpha : [0, \infty) \to M$ which "escapes" every compact set $K \subset M$ (i.e. given K, there exists $t_0 \in (0, \infty)$ such that $\alpha(t)$ is not in K for all $t > t_0$). Define the length of α by
$$\lim_{t\to\infty} \int_0^t |\alpha'(t)| dt.$$
Prove that M is complete if and only if every divergent curve has infinite length.