

## MAT 566 Differential Topology : Exercise Sheet Five

1. (i) For  $n > m$ , show that two arbitrary embeddings  $\mathbb{R}^m \rightarrow \mathbb{R}^n$  are isotopic.  
 (ii) Let  $f : (0, 1) \rightarrow \mathbb{R}^2$  be the embedding which takes  $t$  to  $(t, 0)$ . Give an isotopy of  $f$  which cannot be embedded in a diffeotopy of  $\mathbb{R}^2$ .
2. Let  $f$  and  $g$  be the embeddings of the circle  $S^1$  in  $\mathbb{C} - \{0\}$  given by  $\theta \mapsto e^{i\theta}$  and  $\theta \mapsto 2 + e^{i\theta}$  respectively. Show that  $f$  and  $g$  are not isotopic in  $\mathbb{C} - \{0\}$ .
3. Find an isotopy  $h : \mathbb{R} \times M \rightarrow N$  which is such that the map from  $\mathbb{R} \times M \rightarrow \mathbb{R} \times N$  given by

$$(t, x) \mapsto (t, h(t, x))$$

fails to be an embedding. [Hint: Take  $M = \mathbb{R}$  and  $N = \mathbb{R}^2$ .]

4. Show that the degree of a composition  $g \circ f$  is equal to the product  $(\deg g)(\deg f)$ .
5. Let  $f$  and  $g$  be two maps from  $X$  to the unit sphere  $S^p$  which satisfy  $\|f(x) - g(x)\| < 2$  for all  $x$  (in other words,  $f(x)$  and  $g(x)$  are never antipodal points). Prove that  $f$  is homotopic to  $g$ .
6. For  $m < p$ , show that every map  $M^m \rightarrow S^p$  is homotopic to a constant.
7. (i) Suppose that  $f : S^n \rightarrow S^n$  is a map with degree not equal to  $(-1)^{n+1}$ . Prove that  $f$  has a fixed point.  
 (ii) Suppose that  $f : S^n \rightarrow S^n$  is a map with odd degree. Show that there exists a pair of antipodal points  $\{x, -x\}$  which are carried to antipodal points, ie.

$$\{f(x), f(-x)\} = \{y, -y\}$$

for some  $y \in S^n$ .

8. Let  $M^m$  and  $N^n$  be disjoint submanifolds of  $\mathbb{R}^{k+1}$ . The *linking map*

$$\lambda : M \times N \rightarrow S^k$$

is defined by  $\lambda(x, y) = (x - y)/\|x - y\|$ . If  $M$  and  $N$  are compact and oriented, with  $m + n = k$ , then define the *linking number*  $l(M, N)$  of  $M$  and  $N$  to be the degree of  $\lambda$ .

- (i) Prove that

$$l(M, N) = (-1)^{(m+1)(n+1)}l(N, M).$$

(ii) If there exists an oriented submanifold  $X \subset \mathbb{R}^{k+1}$  with  $M = \partial X$  and  $X$  does not intersect  $N$ , then show that  $l(M, N) = 0$ .

(iii) Define the linking number for submanifolds of the sphere  $S^{m+n+1}$ .

9. Let  $y$  and  $z$  be distinct regular values for the smooth map  $f : S^{2p-1} \rightarrow S^p$ . The smooth manifolds  $f^{-1}(y)$  and  $f^{-1}(z)$  can be given orientations (as in our proof that the Brouwer degree is a homotopy invariant). Hence the linking number  $l(f^{-1}(y), f^{-1}(z))$  from Exercise 7(iii) is well-defined.

(i) Prove that  $l(f^{-1}(y), f^{-1}(z))$  is a locally constant function of (regular values)  $y$ .

(ii) Suppose we have another map  $g : S^{2p-1} \rightarrow S^p$  such that  $y$  and  $z$  are also regular values of  $g$ , and

$$\|f(x) - g(x)\| < \|y - z\|$$

for all  $x$ . Prove that

$$l(f^{-1}(y), f^{-1}(z)) = l(g^{-1}(y), f^{-1}(z)) = l(g^{-1}(y), g^{-1}(z)).$$

(iii) Prove that  $l(f^{-1}(y), f^{-1}(z))$  depends only on the homotopy class of the map  $f$ , and moreover does not depend on the choice of distinct regular values  $y$  and  $z$ .

[The integer  $H(f) := l(f^{-1}(y), f^{-1}(z))$  is called the *Hopf invariant* of  $f$ .]

10. (i) If  $p$  is odd, prove that  $H(f)$ , as defined in Exercise 8, must vanish.

(ii) For a composition

$$S^{2p-1} \xrightarrow{f} S^p \xrightarrow{g} S^p$$

prove that

$$H(g \circ f) = H(f)(\deg g)^2.$$

(iii) Think of  $S^3$  as a submanifold of  $\mathbb{C}^2 \cong \mathbb{R}^4$ , and think of  $S^2$  as the complex projective line  $\mathbb{C}P^1$ . The *Hopf fibration*  $\pi : S^3 \rightarrow S^2$  is defined by

$$\pi(z_1, z_2) = [z_1, z_2].$$

Prove that  $H(\pi) = 1$ .

11. Let  $X \subset M$  be a submanifold whose normal bundle possesses a section which is everywhere non-vanishing. Show that the inclusion  $X \subset M$  is isotopic to an embedding whose image does not intersect  $X$ .

12. Let  $M$  be a connected manifold and  $X \subset M$  a codimension one connected submanifold. Show that if the normal bundle  $NX$  is *not* trivial, then  $M - X$  is connected.

13. Let  $M$  be a manifold and  $X \subset M$  a *subset*. Suppose that there exists an open neighbourhood  $U$  of  $X$  and a smooth map  $f : U \rightarrow U$  with  $f \circ f = f$  and  $f(U) = X$ . Prove that  $X$  is a submanifold.