## MAT 566 Differential Topology : Exercise Sheet Two

1. Suppose $E$ admits a Riemannian metric and $F \subset E$ is a subbundle. Prove that the composition
$$F^{\perp} \subset E \stackrel{\text{projection}}{\longrightarrow} E/F$$
is a bundle isomorphism $F^{\perp} \cong E/F$.

2. (i) Let $(E, \pi, X)$ be a vector bundle and $\tilde{\pi} : \tilde{X}(E) \to X$ its orientation cover. Show that $\tilde{\pi}^* E$ has a canonical orientation.

   (ii) By the orientation cover $\tilde{M} \to M$ of a manifold $M$ we mean the orientation cover of $TM$. Show that $\tilde{M}$ is orientable, regardless of whether or not $M$ is.

3. Show that $\mathbb{R}P^n$ is orientable for odd values of $n$ and non-orientable for even values of $n$.

4. A vector bundle is *stably trivial* if its Whitney sum with a suitable trivial bundle is trivial. Show that $TS^n$ is stably trivial.

5. Let $E$ be a rank $k$ vector bundle over $X$ and $\mathcal{U}$ a bundle atlas for $E$ such that all transition functions take values in $\mathrm{O}(n) \subset \mathrm{GL}(k, \mathbb{R})$. Prove there is a unique Riemannian metric $\langle\ ,\ \rangle$ on $E$ such that all charts of $\mathcal{U}$ are isometries on the fibres. (This means that for each bundle chart $h : \pi^{-1}(U) \to U \times \mathbb{R}^k$, the restriction to a fibre $h|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \{x\} \times \mathbb{R}^k$ preserves distances, where $\mathbb{R}^k$ is given the canonical metric.)

6. Let $X$ be a manifold and assume that $X$ admits partitions of unity, subordinate to any open cover. Show that for every line bundle (rank one bundle) $L$ over $X$, $L \otimes L$ is trivial. Is the Whitney sum $L \oplus L$ also trivial?

7. Suppose $M$ is compact, and $f : M \to N$ is a smooth immersion which is globally injective. Prove that $f$ is a smooth embedding.

8. (i) Let $\mathbb{R} + \mathbb{R}$ be the disjoint union of two copies of the real line. Let $f : \mathbb{R} + \mathbb{R} \to \mathbb{R}^2$ be given by $f(x) = (x, 0)$ for $x$ in the first copy of $\mathbb{R}$ and $f(y) = (0, \exp(y))$ for $y$ in the second copy of $\mathbb{R}$. Show that $f$ is an immersion which is globally injective, but not an embedding.

   (ii) As in the first part, let $\mathbb{R} + S^1$ be the disjoint union of the real line and a circle. Define $f : \mathbb{R} + S^1 \to \mathbb{C}$ by $f(x) = (1 + \exp(x)) \exp(ix)$ for $x \in \mathbb{R}$ and $f(\exp(it)) = \exp(it)$ for $\exp(it) \in S^1$. Prove that $f$ is an immersion which is globally injective, but not an embedding.

   [It may help to sketch the images of these maps.]

9. Define the map $f : \mathbb{R} \to \mathbb{C} \times \mathbb{C}$ by

$$t \mapsto (\exp(ait), \exp(bit))$$

for real numbers $a$ and $b$. Show that $f$ is an immersion provided $a$ or $b$ is nonzero. Assume $b \neq 0$ and $a/b$ is irrational; show that $f$ is globally injective and the image is dense in $S^1 \times S^1 \subset \mathbb{C}^2$. Is $f$ an embedding?

10. Let $A$ be a symmetric real $n \times n$ matrix, and $b \in \mathbb{R}$. Define the *quadric* by

$$M := \{x \in \mathbb{R}^n \mid \ {}^t x A x = b\}.$$

Is $M$ a submanifold of $\mathbb{R}^n$? Does the answer depend on how $A$ and/or $b$ are chosen?

11. Define the smooth map

$$f : \mathbb{R}^{n^2} \to S$$

from the space of $n \times n$-matrices to the space of symmetric matrices by $f(A) := \ {}^t AA$. Complete the proof that the *orthogonal group* $\mathrm{O}(n)$ of real orthogonal $n \times n$-matrices is a smooth submanifold of $\mathbb{R}^{n^2}$ by showing that the identity matrix is a regular value of $f$.

12. A $k$-frame in $\mathbb{R}^n$ is an *orthonormal* $k$-tuple $(v_1, \ldots, v_k)$ of vectors in $\mathbb{R}^n$. The set $V_n^k$ of $k$-frames in $\mathbb{R}^n$ is called a *Stiefel manifold*. Show that $V_n^k$ is a compact smooth manifold of dimension $nk - k(k+1)/2$.

13. Show that there is an immersion of the punctured torus $S^1 \times S^1 \backslash \{\text{a point}\}$ into $\mathbb{R}^2$. [Hint: Spread out the puncture.] Is there an embedding of the punctured torus in $\mathbb{R}^2$?

14. Let $M$ be a compact manifold of dimension $n \geq 1$ and let $f : M \to \mathbb{R}$ be a smooth map. Prove that $M$ has at least two critical points.

15. For an integer $d \geq 0$, $W^{2n-1}(d)$ is defined as the set of points $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$ such that

$$z_0^d + z_1^2 + z_2^2 + \ldots + z_n^2 = 0$$

and

$$z_0 \bar{z}_0 + z_1 \bar{z}_1 + \ldots + z_n \bar{z}_n = 2.$$

Show that $W^{2n-1}(d)$ is a smooth manifold of dimension $2n - 1$. [These examples are known as *Brieskorn manifolds*.]