MAT 566 Differential Topology : Exercise Sheet One

1. Using stereographic projection, show that the sphere S^n possesses a smooth atlas with precisely two coordinate charts.

2. (i) Let M be a smooth manifold and $\sigma : M \to M$ a fixed point free involution. In other words, σ is a diffeomorphism such that $\sigma \circ \sigma = \mathrm{Id}_M$ and $\sigma(x) \neq x$ for all $x \in M$. Show that the quotient space M/σ can be given a smooth structure such that $M \to M/\sigma$ is a local diffeomorphism, and this smooth structure is unique. (An example is the antipodal map acting on S^n, with quotient real projective space $\mathbb{R}P^n$.)

   (ii) Given an example where σ is *not* fixed point free, and the quotient is *not* a manifold.

   (iii) Show that $\mathbb{R}P^1$ is diffeomorphic to S^1.

   (iv) Show that $\mathbb{R}P^2$ is obtained by gluing a disc D^2 and a Möbius band along their common boundary S^1. (A heuristic argument will suffice - later we will study manifolds with boundary and how to smoothly glue them.)

3. Prove that $\mathbb{R}^k$ can be embedded in an n-dimensional (topological or smooth) manifold M if $k \leq n$.

4. (i) Show that the Cartesian product $f_1 \times f_2 : N_1 \times N_2 \to M_1 \times M_2$ of two embeddings is an embedding.

   (ii) Show that if the n-dimensional manifold M is a product of spheres, then it can be embedded in $\mathbb{R}^{n+1}$. For example, the torus $S^1 \times S^1$ can clearly be embedded in $\mathbb{R}^3$. (Hint: First describe an embedding $S^n \times \mathbb{R} \to \mathbb{R}^{n+1}$, then use part (i).)

5. Does there exist a smooth map $f : N \to M$ such that $f : N \to f(N)$ is a diffeomorphism but $f(N) \subset M$ is *not* a smooth submanifold?

6. (i) Let $\mathcal{E}(p)$ be the algebra of function germs at a point $p \in M$. Show that the set of derivations of $\mathcal{E}(p)$ forms a vector space.

   (ii) Let $f : N \to M$ be an embedding with $f(p) = q$. Show that $f^* : \mathcal{E}(q) \to \mathcal{E}(p)$ is surjective and $T_p f$ is injective.

7. For $n > 1$ let $M = \{x \in \mathbb{R}^n | x_1^2 = x_2^2 + \ldots + x_n^2 \text{ and } x_1 \geq 0\}$. Show that M is *not* a smooth submanifold of $\mathbb{R}^n$.

8. Let E be a vector bundle over a connected manifold X, and let $f : E \to E$ be a bundle homomorphism such that $f \circ f = \mathrm{Id}_E$. Show that

$$\mathrm{Fix}(f) := \{v \in E | f(v) = v\}$$

   is a subbundle of E. If E has rank $2k$, can you predict the rank of $\mathrm{Fix}(f)$?

9. Let E and E' be the trivial rank two and rank one bundles over $\mathbb{R}^2$, respectively. Define a smooth map $f : E \to E'$ by

$$\begin{array}{rccc} f_{(x,y)} & : & E_{(x,y)} & \to & E'_{(x,y)} \\ & & (x,y,\mu,\nu) & \mapsto & (x,y,x\mu + y\nu) \end{array}$$

where $(x,y) \in \mathbb{R}^2$. Describe kernel$f$. Is it a vector bundle in a neighbourhood of the origin $(0,0) \in \mathbb{R}^2$?

10. (i) Let E be a vector bundle over X, and let $X_0 \subset X$ be a submanifold of X, with $i : X_0 \to X$ the inclusion. Show that i^*E and $E|_{X_0}$ are canonically isomorphic.

    (ii) Show that if E is a trivial bundle over X, and $f : Y \to X$, then the induced bundle f^*E over Y is also trivial.

11. (i) Recall that $\mathbb{R}P^n$ is the quotient of S^n by the antipodal map. Show that

$$\eta_n := \{([x], \lambda x) | x \in S^n, \lambda \in \mathbb{R}\}$$

is a *non-trivial* rank one subbundle of the trivial bundle $\mathbb{R}P^n \times \mathbb{R}^{n+1}$. (Hint: If it were trivial, $\eta_n - \{\text{zero-section}\}$ would consist of two connected components.)

    (ii) Prove that every rank one bundle on S^1 is either trivial or isomorphic to the bundle η_1. (Recall that S^1 is diffeomorphic to $\mathbb{R}P^1$.)

12. (i) Show that the tangent bundle of S^2 possesses an atlas with two bundle charts.

    (ii) Construct a vector field on S^2 with exactly two zero points.

    (iii) Construct a vector field on S^2 with exactly one zero point.

13. Let

$$E := \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} | z_0^2 + \ldots + z_n^2 = 1\},$$

regarded as a submanifold of $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$. Show that E is diffeomorphic to the total space of the tangent bundle of S^n.

14. Let s_1 and s_2 be symmetric, positive definite, bilinear forms on a vector space V. Show that the linear combination $(1-t)s_1 + ts_2$ for $t \in [0,1]$ is again a symmetric, positive definite, bilinear form.

15. (i) Suppose $E_1 \cong E_2 \oplus E_3$. Prove that if two of the vector bundles E_i are orientable, then the third one is too.

    (ii) Let E be orientable, and $F \subset E$ a subbundle. Show that E/F is orientable if and only if F is.

    (iii) Let E be an arbitrary vector bundle. Show that $E \oplus E$ is orientable.