

## MAT 542 Complex Analysis I : Exercise Sheet Seven

1. As in the proof of the Riemann Mapping Theorem, let

$$h(z) := \sqrt{\frac{f(z) - w}{1 - \bar{w}f(z)}}$$

and

$$g(z) := \frac{|h'(a)|}{h'(a)} \frac{h(z) - h(a)}{1 - \overline{h(a)}h(z)}.$$

Prove that  $g'(a) > f'(a)$ .

2. Let  $\Omega \subsetneq \mathbb{C}$  be a simply connected region which is equal to its conjugate  $\bar{\Omega}$ , and let  $0 \in \Omega$ . By the Riemann Mapping Theorem there is a bijective conformal mapping  $f : \Omega \rightarrow B(0; 1)$  with  $f(0) = 0$  and  $f'(0) > 0$ . Prove that  $f(\bar{z}) = \overline{f(z)}$ .
3. Let  $\Omega \subsetneq \mathbb{C}$  be a simply connected region with  $0 \in \Omega$ , and  $z \in \Omega$  iff  $-z \in \Omega$ . By the Riemann Mapping Theorem there is a bijective conformal mapping  $f : \Omega \rightarrow B(0; 1)$  with  $f(0) = 0$  and  $f'(0) > 0$ . Prove that  $f(-z) = -f(z)$ .
4. Let  $f : G_1 \rightarrow G_2$  be a homeomorphism (continuous bijection with continuous inverse) between two open sets in  $\mathbb{C}$ . Let  $\{z_n\}$  be a sequence in  $G_1$  converging to a point  $z$  in the boundary  $\partial G_1$ , and suppose that  $\{f(z_n)\}$  converges to  $w$ . Prove that  $w$  must lie on the boundary  $\partial G_2$ .
5. Let  $G_1$  and  $G_2$  be two simply-connected regions, neither one the whole complex plane, and let  $f : G_1 \rightarrow G_2$  be an analytic bijection taking some fixed point  $z_0 \in G_1$  to  $w_0 \in G_2$ . Suppose that  $g : G_1 \rightarrow G_2$  is analytic, one-to-one, and also takes  $z_0$  to  $w_0$ . Prove that  $|g'(z_0)| \leq |f'(z_0)|$ . Does equality imply that  $g$  is onto? Does the inequality still hold if  $g$  is *not* one-to-one? Prove or find a counter-example.
6. Find a bijective conformal mapping from the unit disk  $B(0; 1)$  to the horizontal strip

$$\Omega := \{z \mid -1 < \operatorname{Im} z < 1\}.$$

[Hint: Try using a special case of the Schwarz-Christoffel Formula.]

7. Find a bijective conformal mapping from the upper half-plane  $H = \{z \mid \operatorname{Im} z > 0\}$  to the horizontal half-strip

$$\Omega := \{z \mid \operatorname{Re} z > 0 \text{ and } -1 < \operatorname{Im} z < 1\}.$$

[Hint: Try using a special case of the Schwarz-Christoffel Formula.]

8. Give a direct proof that

$$F(z) = \int_0^z (1 - w^n)^{-2/n} dw$$

is a bijective conformal mapping from  $B(0; 1)$  to the interior of a regular  $n$ -gon.

9. i) If  $u$  is harmonic, prove that

$$u_x = \frac{\partial u}{\partial x} \quad \text{and} \quad u_y = \frac{\partial u}{\partial y}$$

are also harmonic, and that  $f = u_x - iu_y$  is analytic.

ii) If  $f$  is analytic and never zero, prove that  $u = \log|f|$  is harmonic.

10. Let  $f$  be an analytic function on  $G$  and  $u$  a harmonic function on  $f(G)$ . Give a direct proof, using the Chain Rule and Cauchy-Riemann equations for  $f$ , that  $u \circ f$  is harmonic.

11. Let  $u : B(0; 1) \rightarrow \mathbb{R}$  be defined by

$$u(z) := \operatorname{Im} \left[ \left( \frac{z+1}{z-1} \right)^2 \right].$$

Prove that  $u$  is harmonic and  $\lim_{r \rightarrow 1^-} u(re^{i\theta}) = 0$  for all  $\theta$ . Why does this *not* violate the Maximum Principle?

12. Let  $K$  be a compact subset of a region  $G$ . Prove that there is a constant  $C$  depending only on  $K$  and  $G$  such that if  $u$  is any positive harmonic function on  $G$  then  $u(z) \leq Cu(z')$  for every pair of point  $z$  and  $z'$  in  $K$ .

13. Let  $u(x, y)$  be a function on some region  $G$ , and write  $U(r, \theta) := u(r\cos\theta, r\sin\theta)$  for the same function in polar coordinates.

i) Show that the Laplacian looks like

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{r^2} \left[ r \frac{\partial}{\partial r} \left( r \frac{\partial U}{\partial r} \right) + \frac{\partial^2 U}{\partial \theta^2} \right]$$

in polar coordinates.

ii) Solve Laplace's equation for a spherically symmetric function, i.e. assume  $U(r)$  is a function of  $r$  only and solve the PDE above.

14. Let  $a$  be some point in an open region  $G$ , and suppose that  $u$  is harmonic on  $G - \{a\}$  and that  $\lim_{z \rightarrow a} u(z)$  exists and equals  $A$ . Show that we can extend  $u$  to a *harmonic* function on  $G$  by defining  $u(a) = A$ .