MAT 542 Complex Analysis I : Exercise Sheet Five

1. The following functions have isolated singularities at $z = 0$. Determine the nature of each singularity. If it is removable, define $f(0)$. If it is a pole, find the singular part of the Laurent series. If it is essential determine the image of a small anulus $\{z \,|\, 0 < |z| < \delta\}$ under f.

   a) $f(z) = z^{-2}\log(z+1)$ b) $f(z) = z\cos(z^{-1})$

   c) $f(z) = (1-e^z)^{-1}$ d) $f(z) = z^n\sin(z^{-1})$

2. Find all singularities of $f(z) = (\sin(z^{-1}))^{-1}$. Which are isolated and which are not?

3. A function f defined on $\{z \,|\, |z| > R\}$, $R > 0$, has a removable singularity, a pole, or an essential singularity at infinity if $f(z^{-1})$ has the same.

   a) Prove that an entire function with a removable singularity at infinity must be a constant.

   b) Prove that an entire function with a pole of order m at infinity must be a polynomial of degree m.

4. Suppose that f is analytic in the punctured unit disk $B(0,1) - \{0\}$, except for a sequence of poles $\{a_n\}$ which converge to zero. Let λ be an complex number. Show that there is a sequence $\{z_n\}$ of complex numbers which converge to zero and such that $\lim f(z_n) = \lambda$.

5. Let $\lambda \in \mathbb{C}$. Show that

$$\exp\left[\frac{\lambda}{2}\left(z + \frac{1}{z}\right)\right] = a_0 + \sum_{n=1}^{\infty}\left(z^n + \frac{1}{z^n}\right)$$

   for $0 < |z| < \infty$, where

$$a_n = \frac{1}{\pi}\int_0^{\pi} e^{\lambda\cos t}\cos nt\, dt.$$

6. Let $\lambda > 1$. Show that $\lambda - z - e^{-z} = 0$ has exactly one solution in $\{z = x + iy \,|\, x > 0\}$, and this solution must be real. What happens as $\lambda \to 1$?

7. [The Minimum Principle] Suppose f is a non-constant analytic function on a bounded open set G, and continuous on the closure $\bar{G}$. Prove that either f has a zero in G or $|f|$ attains its minimum on ∂G.

8. Let $0 < r < R$ and put $A = \{z \,|\, r \le |z| \le R\}$. Show that there is a positive number $\epsilon > 0$ such that for every polynomial p,

$$\sup\{|p(z) - z^{-1}| \,|\, z \in A\} \ge \epsilon.$$

   This says that z^{-1} is not the uniform limit of polynomials on A.

9. If $f : G \to \mathbb{C}$ is analytic except for poles, show that the set of poles of f cannot have a limit point in G. Find an example with a limit point on the boundary ∂G.

10. Let f be analytic on the open unit disk $B(0,1)$, continuous on $\overline{B(0,1)}$, and $|f(z)| = 1$ whenever $|z| = 1$. Prove that either f is constant or it has a zero in $B(0,1)$.

11. Suppose $f : B(0,1) \to \mathbb{C}$ satisfies $\mathrm{Re} f(z) \geq 0$ for all z in $B(0,1)$ and suppose that f is analytic and not constant. Show that $\mathrm{Re} f(z) > 0$ for all $z \in B(0,1)$.

12. Suppose $|f(z)| \leq 1$ for $z \in B(0,1)$ and f is a non-constant analytic function. By considering the function $g : B(0,1) \to B(0,1)$ defined by

$$g(z) = \frac{f(z) - a}{1 - \bar{a} f(z)}$$

where $a = f(0)$, prove that

$$\frac{|f(0)| - |z|}{1 + |f(0)| \cdot |z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)| \cdot |z|}$$

for $|z| < 1$.

13. a) Does there exist an analytic function $f : B(0,1) \to B(0,1)$ with $f(1/2) = 3/4$ and $f'(1/2) = 2/3$? If so, find such an f. Is it unique?

    b) Does there exist an analytic function $f : B(0,1) \to B(0,1)$ with $f(0) = 1/2$ and $f'(0) = 3/4$? If so, find such an f. Is it unique?

14. Suppose f is analytic in a region containing $\overline{B(0,1)}$ and $|f(z)| = 1$ when $|z| = 1$. Suppose that f has a zero at $z = (1+i)/4$ and a double zero at $z = 1/2$. Can $f(0) = 1/2$?

15. If $f(z)$ is analytic and $\mathrm{Im} f(z) \geq 0$ for $\mathrm{Im} z > 0$, show that

$$\frac{|f(z) - f(z_0)|}{|f(z) - \overline{f(z_0)}|} \geq \frac{|z - z_0|}{|z - \bar{z}_0|}$$

and

$$\frac{|f'(z)|}{\mathrm{Im} f(z)} \leq \frac{1}{y} \qquad \text{where } z = x + iy.$$

16. Show that $|f(z)| \leq 1$ for $|z| \leq 1$ implies

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

17. Let f be analytic on the upper half-plane H, and $|f(z)| \leq 1$ for all $z \in H$. Find a sharp upper bound for $|f'(i)|$, and find all functions which give equality.

18. Let f be analytic on $\mathbb{C}$ except for isolated singularities a_1, a_2, …, a_n. Define the residue of f at infinity to be the residue of $-z^{-2} f(z^{-1})$ at zero. [Why would we define it in this way?] Prove that

$$\mathrm{Res}(f; \infty) + \sum_{k=1}^{n} \mathrm{Res}(f; a_k) = 0.$$

19. Calculate the following integrals:

a) $\int_0^\infty \frac{x}{(x^2+1)(x^2+2)}dx$ b) $\int_0^\infty \frac{\cos x-1}{x^2}dx$

c) $\int_0^\pi \frac{1}{(a+\cos\theta)^2}d\theta$ where $a > 1$

20. Calculate the following integrals:

a) $\int_{-\infty}^\infty \frac{e^{x/2}}{1+e^x}dx$ b) $\int_0^\infty \frac{\log x}{(1+x^2)^2}dx$

c) $\int_0^\infty \frac{\cos x}{(1+x^2)^2}dx$

21. Let γ be the rectangle from $n+\frac{1}{2}+ni$ to $-n-\frac{1}{2}+ni$ to $-n-\frac{1}{2}-ni$ to $n+\frac{1}{2}-ni$ then back to $n+\frac{1}{2}+ni$. Evaluate

$$\int_\gamma \frac{\pi\cot\pi z}{(z+a)^2}dz$$

where a is not an integer. Show that as $n \to \infty$ the integral tends to zero. Deduce that

$$\frac{\pi^2}{\sin^2\pi a} = \sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2}.$$

Finally, by substituting $a = \frac{1}{2}$, show that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

22. Show that a single-valued analytic branch of $\sqrt{1-z^2}$ can be defined in any region such that the points ± 1 are in the same component of the complement. What are the possible values of

$$\int_\gamma \frac{dz}{\sqrt{1-z^2}}$$

over a closed curve in the region?