MAT 542 Complex Analysis I : Exercise Sheet Four

1. Let γ be a simple close curve in $\mathbb{C}$ (i.e. without self-intersections). Use winding numbers to show that the complement of γ consists of at least two connected components. [The Jordan curve theorem says the number of components is exactly two.]

2. Find the power series expansion of $\log z$ around $z = i$ and its radius of convergence.

3. Find the power series expansion of $\sqrt{z}$ around $z = 1$ and its radius of convergence.

4. a) Prove Abel's Theorem: If $\sum a_n z^n$ has radius of convergence 1 and $\sum a_n$ converges to A, then
$$\lim_{r \to 1^-} \sum a_n r^n = A.$$
b) Use part a) to prove that $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \ldots$

5. Find
a) $\int_{|z|=1} z^{-n} \cos z dz, \qquad n \geq 0,$
b) $\int_{|z-1|=1/2} z^{-n} \log z dz, \qquad n \geq 0,$
c) $\int_{|z|=1} z^{-n}(e^z - e^{-z}) dz, \qquad n \geq 0,$
d) $\int_{|z|=2} (z^2+1)^{-1} dz,$
e) $\int_{|z-1|=1/2} z^{1/n}(z-1)^{-n} dz, \qquad n \geq 0.$

6. Find
$$\int_{|z|=r} \frac{z^2+1}{z(z^2+4)} dz$$
where $r > 0$, $r \neq 2$. Does the integral depend on whether $r < 2$ or $r > 2$?

7. Show that
$$f(z) = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right).$$
is a branch of $\arctan z$, and it has power series
$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}$$
for $|z| < 1$.

8. Show that the power series for $\sec z$ has the form
$$\sec z = 1 + \sum_{n=1}^{\infty} \frac{E_{2n}}{(2n)!} z^{2n}.$$

Find E_2, E_4, E_6, E_8, and the radius of convergence. The E_{2n} are known as Euler's numbers. Show that

$$E_{2n} - \binom{2n}{2n-2} E_{2n-2} + \binom{2n}{2n-4} E_{2n-4} - \ldots + (-1)^{n-1}\binom{2n}{2} E_2 + (-1)^n = 0.$$

[Hint: Use the fact that $(\cos z)(\sec z) = 1$.]

9. Find the power series of $\frac{e^z-1}{z}$ about $z = 0$ and its radius of convergence. Suppose that $f(z) = \frac{z}{e^z-1}$ has power series

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n.$$

What is the radius of convergence? Show that

$$a_0 + \binom{n+1}{1} a_1 + \ldots + \binom{n+1}{n} a_n = 0.$$

Show that $a_n = 0$ for n odd and greater than one. [Hint: $f(z) + z/2$ is an even function.] Find a_2, a_4, a_6, a_8, and a_{10}. Up to a sign, these are known as the Bernoulli numbers.

10. Let f be an entire function. Suppose that $|f(z)| \leq M|z|^n$ for $|z| > R$, for some constant M, $R > 0$, and integer $n \geq 1$. Show that f is a polynomial of degree at most n.

11. Find all entire functions f such that $f(x) = e^x$ for $x \in \mathbb{R}$.

12. Let $f : G \to \mathbb{C}$ be analytic in a region G. Suppose $a \in G$ is a global minimum, i.e. $|f(a)| \leq |f(z)|$ for all $z \in G$. Prove that either $f(a) = 0$ or f is constant.