MAT 542 Complex Analysis I : Comps Problems

1. [Jan 1990] Find a one-to-one holomorphic map from the unit disk $\{|z| < 1\}$ onto the slit disk $\{|z| < 1\} - \{[0, 1)\}$. [Hint: It suffices to find a sequence of one-to-one holomorphic maps whose composition does this.]

2. [May 1990] Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a positive harmonic function. Prove that u must be constant.

3. [May 1990] Prove that an entire function $f : \mathbb{C} \to \mathbb{C}$ has an inverse if and only if $f(z) = az + b$ with $a \neq 0$.

4. [May 1990] Let D be a domain in $\mathbb{C}$. A function $f : D \to \mathbb{C}$ is *schlicht* or *univalent* if f is both holomorphic and one-to-one in the domain D. For which values of the complex parameter λ is the function $f_\lambda(z) = z + \lambda z^2$ univalent in the open unit disk?

5. [Jan 1991] Let H be the upper half-plane

$$H = \{z \in \mathbb{C} \,|\, \mathrm{Im} z > 0\}.$$

Let z_0, z_1 be points in H. Determine $\sup|f'(z_0)|$, where the supremum is taken over all functions f, holomorphic in H, with $f(H) \subset H$ and $f(z_0) = z_1$.

6. [Jan 1991] Let f be a fractional linear transformation of the Riemann sphere which maps the unit disk *onto* itself, with $f(0) \neq 0$. Show that $f(\infty) \neq \infty$, and $f(0) \cdot \overline{f(\infty)} = 1$.

7. [May 1991] Let A be the $n \times n$ complex matrix

$$A = \begin{pmatrix} \lambda_0 & 1 & 0 & \cdots & 0 \\ 0 & \lambda_0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ & & & \lambda_0 & 1 \\ 0 & \cdots & & 0 & \lambda_0 \end{pmatrix},$$

having λ_0 on the diagonal and 1 just above the diagonal (Jordan form). Then the function

$$z \mapsto (A - zI)^{-1}$$

can be viewed as a matrix-valued function defined on $\mathbb{C}\backslash\{\lambda_0\}$. Let $\gamma \subset \mathbb{C}\backslash\{\lambda_0\}$ be a simple, smooth, closed curve, counter-clockwise oriented. Define

$$P_A = \frac{1}{2\pi i} \int_\gamma (A - zI)^{-1} dz.$$

Show that

$$-P_A^2 = P_A.$$

[Hint: Note that $A = \lambda_0 I + N$ where N is nilpotent.]

8. [May 1991] For a complex variable z, define $\sin z$ by

$$2i \sin z = e^{iz} - e^{-iz}.$$

Find all zeros of $\sin z$.

9. [Jan 1992] Evaluate

$$\int_\gamma z^{\frac{1}{2}}(1+z)^{\frac{1}{2}} dz$$

where $\gamma : [0,1] \to \mathbb{C}$, $\gamma(t) = -e^{-2\pi it}$, and we use the principal branch of the power functions.

10. [Aug 1992] Let D_r be the disk of radius r in $\mathbb{C}$ centred at 0. Let $M > 0$. Suppose f is holomorphic on D_1, $f(0) = 0$, and $\mathrm{Re} f \leq M$ on D_1. Show that $|f| \leq 2M$ on $D_{1/2}$. [Hint: Consider $g(z) = f(z)/(2M - f(z))$; what can you say about $|g|$ on D_1? on $D_{1/2}$?]

11. [Aug 1992] Given N points on a circle of radius 1, show that there is another point on the circle, the product of whose distances from the given points is at least 1. [Hint: You may wish to apply a theorem from complex analysis to an appropriate holomorphic function on the unit disk.]

12. [Jan 1993] Let f be a harmonic function on the unit disk $D = \{z \,|\, |z| < 1\}$. Suppose that f is positive (i.e. that $f(z) > 0$ for all $z \in D$), and that for almost all $\theta \in [0, 2\pi]$, $g(\theta) = \lim_{r \to 1^-} f(re^{i\theta})$ exists.

a) Show that $\frac{1}{2\pi} \int_0^{2\pi} g(\theta) d\theta \leq f(0)$.

b) Given an example in which equality fails to hold in part a).

[Hint for b): Conformal mappings can lead to examples of positive harmonic functions.]

13. [Jan 1993] Let D be the sector $\{z \,|\, -\pi/4 < \arg(z) < \pi/4\}$. Find all fractional linear transformations mapping D onto itself.

14. [Aug 1993] Let $p(z)$ be a complex polynomial of degree $d \geq 2$. Assume that all roots a_j, $j = 1, \ldots, d$ of the equation $p(z) = 0$ are simple. Let $\lambda_j = p'(a_j)$. Prove that

$$\sum_{j=1}^{d} \frac{1}{\lambda_j} = 0.$$

15. [Aug 1993] Let S denote the sector in the complex plane given by

$$S = \{z \,|\, -\pi/4 < \arg(z) < \pi/4\}.$$

Let $\bar{S}$ denote the closure of S. Let f be a continuous complex function on $\bar{S}$ which is holomorphic on S. Suppose further:

i) $|f(z)| \leq 1$ for all z in the boundary of S;

ii) $|f(x+iy)| \leq e^{\sqrt{x}}$ for all $x + iy \in S$.

Prove that $|f(z)| \leq 1$ for all $z \in S$. [Hint: You may wish to first prove that for each $\epsilon > 0$, $|f(x+iy)| \leq e^{\epsilon x}$.]

16. [Aug 2002] Let D be the domain in the extended complex plane $\mathbb{C} \cup \{\infty\}$ exterior to the circles $|z-1| = 1$ and $|z+1| = 1$. Find a Riemann map (one-to-one holomorphic map) of D onto the strip $S = \{z \in \mathbb{C} \,|\, 0 < \mathrm{Im} z < 2\}$.

17. [Aug 2002] Let H be the upper half-plane, and $f : H \to H$ an antiholomorphic map such that $f \circ f = \mathrm{Id}$.

    a) Show that f can be written as $f(z) = \overline{g(z)}$ where

    $$g(z) = \frac{az+b}{cz+d}, \qquad a, b, c, d \text{ real}, \qquad ad - bc = -1.$$

    b) Show that f is conjugate to $f_0 : z \mapsto -\bar{z}$, i.e.

    $$f = \phi \circ f_0 \circ \phi^{-1}$$

    for some biholomorphic map $\phi : H \to H$.

18. [Aug 2002] a) Let Ω be a connected and simply-connected region in $\mathbb{C}$. Show that for any real-valued funcion u which is harmonic in Ω (i.e. $u \in \mathcal{C}^2(\Omega)$, $u_{xx} + u_{yy} = 0$ in Ω, with subscripts denoting partial derivatives), there exists a real-valued function v such that $u + iv$ is holomorphic in Ω.

    b) Give an example which shows that the conclusion of part a) can fail if Ω is not simply-connected.

19. [Jan 2003] Let f be a holomorphic function on the punctured plane $0 < |z| < \infty$. Assume that there exists a positive constant C and a real constant N such that

    $$|f(z)| \leq C|z|^N \text{ for } 0 < |z| < \frac{1}{2}.$$

    Show that $z = 0$ is either a pole or a removable singularity for f and find a bound for $\mathrm{ord}_0 f$, the order of the function f at 0.

20. [Jan 2003] a) State any form of Picard's Theorem.

    b) Let $n \geq 2$ be an integer. Show that there are no nowhere vanishing and nonconstant entire functions f and g that satisfy

    $$f^n + g^n = 1.$$

    c) Assume now that $n > 2$. What are all the solutions f and g in the ring of entire functions that satisfy

    $$f^n + g^n = 1?$$

    [Hint: Transform the problem to the setting of meromorphic functions on the complex line.]

21. [Jan 2003] Suppose f is a meromorphic function on the plane which is holomorphic on the unit disk and has one simple pole only on the unit circle at the point z_0. Let $\sum_{n=0}^{\infty} a_n z^n$ be the power series for f at the origin. Show that $\lim_{n\to\infty} a_n z_0^n$ exists.

22. [Aug 2003] Prove that for any $r > 0$ there exists $N > 0$ such that for every $n > N$, the polynomials
$$P_n(z) = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!}$$
have no zeros in the disk $\{z \in \mathbb{C} \,|\, |z| < r\}$.

23. [Aug 2003] Let f be a holomorphic function on the unit disk $\{z \in \mathbb{C} \,|\, |z| < 1\}$ such that $|f(z)| < 1$ for $|z| < 1$. If $f\left(\frac{1}{2}\right) = \frac{1}{3}$, find a sharp upper bound for $|f'\left(\frac{1}{2}\right)|$.

24. [Aug 2003] a) Show that
$$\lim_{M\to\infty} \int_{S_M} \frac{e^{iz}}{z} dz = 0$$
where S_M is the semicircle in the upper half plane with center at 0 and radius $M > 0$.

b) Evaluate
$$\int_0^\infty \frac{\sin x}{x} dx.$$

25. [Jan 2004] Suppose $\Omega = \{z \in \mathbb{C} \,|\, |z-1| > 1 \text{ and } |z+1| > 1\}$. Let U be a harmonic function on Ω with boundary values $U = 0$ on $\{|z-1| = 1\}$ and $U = 1$ on $\{|z+1| = 1,\, z \neq 0\}$. Find the value of $U(4)$.

26. [Jan 2004] Let g denote a holomorphic function on the subset of the complex plane given by $|z| < r$, where r is a fixed real number satisfying $r > 1$. Suppose that $|g(z)| \leq 1$ holds for all $|z| \leq 1$.

a) Show that for all $t \in \mathbb{C}$ with $|t| < 1$, the equation
$$z = tg(z)$$
has a unique solution $z = s(t)$ in the disk $|z| < 1$.

b) Show that $t \mapsto s(t)$ is a holomorphic function on the disk $|t| < 1$.

27. [Aug 2004] Let f be a holomorphic function on the open unit disk $D = \{z \in \mathbb{C} \,|\, |z| < 1\}$ such that $f(0) = 0$ and $|f(z) + zf'(z)| < 1$ for all $z \in D$. Show that $|f(z)| \leq |z|/2$ for all $z \in D$.

28. [Aug 2004] Let $f, g \in \mathcal{O}(D) \cap \mathcal{C}(\bar{D})$ where $D \subset \mathbb{C}$ is the open unit disk $\{z \in \mathbb{C} \,|\, |z| < 1\}$. Suppose there is a circle homeomorphism $h : \partial D \to \partial D$ such that $f(t) = (g \circ h)(t)$ for all $t \in \partial D$. Prove that $f(\bar{D}) = g(\bar{D})$; i.e. the two functions have the same range on the closed disk.