Name:

M400D Topics in Mathematics - Topology : Practise Midterm Exam

1. Let x, y, z, and w be any four points in a metric space (X, d). Use the triangle inequality to prove that

$$d(x, z) + d(y, w) \geq |d(x, y) - d(z, w)|.$$

2. Let a and b be two points in a metric space (X, d), and $r > 0$ and $s > 0$ real numbers. Suppose $B(a; r) = B(b; s)$. Is $a = b$? Is $r = s$? Prove or find counterexamples.

3. In a topological space $(X, \mathcal{T})$, a finite intersection $O_1 \cap O_2 \cap \cdots \cap O_n$ of open sets is an open set. Find an example which shows that an infinite intersection $\bigcap_{n\geq 1} O_n$ of open sets need *not* be an open set.

4. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ and $g : (Y, \mathcal{T}') \to (Z, \mathcal{T}'')$ be continuous functions between topological spaces. Prove that the composition

$$g \circ f : (X, \mathcal{T}) \to (Z, \mathcal{T}'')$$

is also continuous.

5. Let X be a set and $\mathcal{T}_1$ and $\mathcal{T}_2$ two topologies on X.

a) Prove that the intersection of $\mathcal{T}_1$ and $\mathcal{T}_2$ is a topology on X. By intersection we mean

$$\mathcal{T}_1 \cap \mathcal{T}_2 = \{O \subset X | O \in \mathcal{T}_1 \text{ and } O \in \mathcal{T}_2\}.$$

b) Prove that the union of $\mathcal{T}_1$ and $\mathcal{T}_2$ might *not* be a topology on X (find an example). By union we mean

$$\mathcal{T}_1 \cap \mathcal{T}_2 = \{O \subset X | O \in \mathcal{T}_1 \text{ or } O \in \mathcal{T}_2\}.$$

6. a) Let $(X, \mathcal{T})$ be a topological space and A_n, $n \geq 1$, be an infinite collection of subsets of X. Prove that

$$\bigcup_{n\geq 1} \overline{A_n} \subset \overline{\bigcup_{n\geq 1} A_n}.$$

b) By considering the subsets $A_n = (\frac{1}{n}, 1)$ of $\mathbb{R}$, show that the left and right hand sides in a) need not be equal.

7. Recall that the boundary of a subset A of a topological space $(X, \mathcal{T})$ is the intersection of the closure of A with the closure of the complement of A,

$$\text{Bdry}(A) = \overline{A} \cap \overline{X - A}.$$

What is the boundary of the subset $A = \{\frac{1}{n} | n \in \mathbb{N}\}$ of $\mathbb{R}$?