Name: Time allowed: 50 minutes.
Calculators are not allowed.

M369 Linear Algebra (section 1) : Solutions to practise for 2nd Midterm Exam

F If $N(A) \neq \{\mathbf{0}\}$ then $A\mathbf{x} = \mathbf{b}$ will have a unique least squares solution.

We can always add a non-zero vector from $N(A)$ to a least squares solution to produce a different least squares solution.

F The vector projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by

$$\mathbf{p} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{u}.$$

The correct formula is

$$\mathbf{p} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v}.$$

T If $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$, then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

The difference $\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2 = 2\langle \mathbf{x}, \mathbf{y} \rangle$ must vanish.

F The vectors $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ form an orthonormal set of vectors in $\mathbb{R}^2$.

They are orthogonal but not orthonormal, as they are not unit vectors.

T Let $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ be an orthonormal basis for $\mathbb{R}^3$ and let $\mathbf{v} \in \mathbb{R}^3$ be a unit vector. If $\mathbf{v}^T\mathbf{u}_1 = 3/5$ and $\mathbf{v}^T\mathbf{u}_2 = 4/5$, then $\mathbf{v}$ must be orthogonal to $\mathbf{u}_3$.

By Parseval's formula

$$\|\mathbf{v}\|^2 = (\mathbf{v}^T\mathbf{u}_1)^2 + (\mathbf{v}^T\mathbf{u}_2)^2 + (\mathbf{v}^T\mathbf{u}_3)^2.$$

Substituting $\|\mathbf{v}\| = 1$, $\mathbf{v}^T\mathbf{u}_1 = 3/5$, and $\mathbf{v}^T\mathbf{u}_2 = 4/5$ shows that $\mathbf{v}^T\mathbf{u}_3 = 0$.

T If $\mathbf{x}$ is an eigenvector of the non-singular matrix A, then $\mathbf{x}$ must also be an eigenvector of the inverse matrix A^{-1}.

We have $A\mathbf{x} = \lambda\mathbf{x}$. Multiplying both sides by $\lambda^{-1}A^{-1}$ gives $A^{-1}\mathbf{x} = \lambda^{-1}\mathbf{x}$. Note that $\lambda \neq 0$, because if 0 were an eigenvalue A would be singular.

F The sum $\lambda_1 + \ldots + \lambda_n$ of the eigenvalues of A must equal the determinant of A.

The product of the eigenvalues is equal to the determinant. The sum of the eigenvalues is equal to the trace of A.

2. Let S be the subspace of $\mathbb{R}^4$ spanned by

$$\begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ -2 \\ 5 \\ -3 \end{pmatrix}.$$

a) Write down a basis for $S^{\perp}$.

We have to find the space of all vectors $\mathbf{x}$ *orthogonal to the two given vectors. Thus we should solve*

$$\begin{pmatrix} 1 & -1 & 2 & -2 \\ 2 & -2 & 5 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \mathbf{0}.$$

The reduced row echelon form is

$$\begin{pmatrix} 1 & -1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and the general solution is

$$\mathbf{x} = \begin{pmatrix} x_2 + 4x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix}.$$

Thus a basis for $S^{\perp}$ *is*

$$[\mathbf{v}_1, \mathbf{v}_2] = \left[\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right].$$

b) Find the distance from $\mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 3 \\ -5 \end{pmatrix}$ to $S^{\perp}$.

The projection of $\mathbf{x}$ *onto* $S^{\perp}$ *is*

$$\mathbf{p} = \frac{\langle \mathbf{x}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 + \frac{\langle \mathbf{x}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 = 1.\mathbf{v}_1 + 0.\mathbf{v}_2.$$

The distance from $\mathbf{x}$ *to* $S^{\perp}$ *is then the length of* $\mathbf{x} - \mathbf{p}$*, which equals*

$$\|\mathbf{x} - \mathbf{p}\| = \| \begin{pmatrix} 2 \\ 0 \\ 3 \\ -5 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \| = \sqrt{1^2 + (-1)^2 + 3^2 + (-5)^2} = 6.$$

3. Find all least squares solutions to the inconsistent system of equations

$$\begin{aligned} x_1 - x_2 &= 0, \\ x_2 - x_3 &= 1, \\ x_3 - x_1 &= 2. \end{aligned}$$

Let

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \qquad \textit{and} \qquad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

Then the normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$ *are*

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}.$$

The reduced row echelon form is

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and thus the least squares solutions are

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 - 1 \\ x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

4. Let

$$A = \begin{pmatrix} 1 & 2 & -4 & 1 \\ -2 & -3 & 5 & 0 \\ 3 & 4 & -6 & -1 \end{pmatrix}.$$

Find bases for $N(A)$, $R(A)$, $N(A^T)$, and $R(A^T)$.

The reduced row echelon forms of A and A^T are

$$U = \begin{pmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \textit{and} \qquad U' = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

respectively. From the first of these, we see that the solution to $A\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{pmatrix} -2x_3 + 3x_4 \\ 3x_3 - 2x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix}.$$

Thus bases for $N(A)$ and the row space $R(A^T)$ of A are

$$\left[\begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right] \qquad \textit{and} \qquad \left[\begin{pmatrix} 1 \\ 0 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \end{pmatrix} \right]$$

respectively. These are orthogonal subspaces of $\mathbb{R}^4$. Similarly, the solution to $A^T\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = \begin{pmatrix} x_3 \\ 2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Thus bases for $N(A^T)$ and the column space $R(A)$ of A are

$$\left[\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right] \qquad \textit{and} \qquad \left[\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right]$$

respectively. These are orthogonal subspaces of $\mathbb{R}^3$.

5. Let A be an $n \times n$ matrix, B an $n \times r$ matrix, and $C = AB$. Show that

a) the null space $N(B)$ is a subspace of the null space $N(C)$,

Let $\mathbf{x}$ be in the null space $N(B)$. Then $B\mathbf{x} = \mathbf{0}$, and therefore $C\mathbf{x} = AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$. So $\mathbf{x}$ is also in the null space $N(C)$. This proves that $N(B)$ is a subspace of $N(C)$.

b) the row space $R(C^T)$ of C is a subspace of the row space $R(B^T)$ of B.

[Hint: Take complements of part a), and use the fact that $N(X)^\perp = R(X^T)$ for a matrix X.]

Let $\mathbf{x}$ be in $N(C)^\perp$. Then

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0$$

for all $\mathbf{y}$ in $N(C)$. In particular

$$\langle \mathbf{x}, \mathbf{y} \rangle = 0$$

for all $\mathbf{y}$ in $N(B)$, since $N(B) \subset N(C)$. Therefore $\mathbf{x}$ is in $N(B)^\perp$. This proves that $N(C)^\perp$ is a subspace of $N(B)^\perp$, but

$$N(C)^\perp = R(C^T)$$

and

$$N(B)^\perp = R(B^T).$$

So in fact we have proved that the row space $R(C^T)$ of C is a subspace of the row space $R(B^T)$ of B.

6. Let

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 2 \end{pmatrix}.$$

a) Use the Gram-Schmidt process to find an orthonormal basis for the column space of A.

Denote the columns of A by $\mathbf{a}_1$ and $\mathbf{a}_2$. Then $r_{11} = \|\mathbf{a}_1\| = \sqrt{3}$. The first step is to normalize $\mathbf{a}_1$ so that it has unit length: we get

$$\mathbf{q}_1 = \frac{\mathbf{a}_1}{r_{11}} = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}.$$

Next $r_{12} = \langle \mathbf{a}_2, \mathbf{q}_1 \rangle = \sqrt{3}$, and we project $\mathbf{a}_2$ onto $\mathbf{q}_1$ to get

$$\mathbf{p}_1 = r_{12}\mathbf{q}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Therefore

$$\mathbf{a}_2 - \mathbf{p}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Now $r_{22} = \|\mathbf{a}_2 - \mathbf{p}_1\| = \sqrt{6}$ and we use this to normalize $\mathbf{a}_2 - \mathbf{p}_1$ so that it has unit length: we get

$$\mathbf{q}_2 = \frac{\mathbf{a}_2 - \mathbf{p}_1}{r_{22}} = \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}.$$

Thus the orthonormal basis is

$$[\mathbf{q}_1, \mathbf{q}_2] = \left[\begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} \right].$$

b) Using your results from part a), factor A into a product QR, where Q has orthonormal column vectors and R is upper triangular.

From part a)

$$Q = (\mathbf{q}_1, \mathbf{q}_2) = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{6} \\ -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix}$$

and

$$R = \begin{pmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{6} \end{pmatrix}.$$

You should check that QR does indeed equal A.

7. a) Find the eigenvalues and corresponding eigenvectors of $A = \begin{pmatrix} 3 & 1 \\ 5 & -1 \end{pmatrix}$.

The characteristic equation is

$$\det \begin{pmatrix} 3-\lambda & 1 \\ 5 & -1-\lambda \end{pmatrix} = (3-\lambda)(-1-\lambda) - 5 = \lambda^2 - 2\lambda - 8 = (\lambda-4)(\lambda+2).$$

Thus the eigenvalues are $\lambda_1 = 4$ *and* $\lambda_2 = -2$.
The 4*-eigenvector must satisfy*

$$\begin{pmatrix} -1 & 1 \\ 5 & -5 \end{pmatrix} \mathbf{x} = \mathbf{0}$$

and the solution is $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, *or any non-zero multiple of this vector.*
Similarly the -2*-eigenvector must satisfy*

$$\begin{pmatrix} 5 & 1 \\ 5 & 1 \end{pmatrix} \mathbf{x} = \mathbf{0}$$

and the solution is $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$, *or any non-zero multiple of this vector.*

b) Suppose that the $n \times n$ matrix A satisfies $A^2 = A$. Show that the only possible eigenvalues of A are 0 and 1.

If λ *is an eigenvalue then there exists a non-zero vector* $\mathbf{x}$ *such that* $A\mathbf{x} = \lambda\mathbf{x}$. *Multiplying both sides by* A *gives*

$$A^2\mathbf{x} = A\lambda\mathbf{x} = \lambda A\mathbf{x} = \lambda^2\mathbf{x}.$$

However, since $A^2 = A$, *we also have*

$$A^2\mathbf{x} = A\mathbf{x} = \lambda\mathbf{x}.$$

Comparing these two equations, we see that $\lambda^2 = \lambda$, *since* $\mathbf{x}$ *is a non-zero vector. Therefore* λ *must be either* 0 *or* 1.