Name: Time allowed: 50 minutes.
Calculators are not allowed.

M369 Linear Algebra (section 1) : Practise for 2nd Midterm Exam

T/F If $N(A) \neq \{\mathbf{0}\}$ then $A\mathbf{x} = \mathbf{b}$ will have a unique least squares solution.

T/F The vector projection of $\mathbf{u}$ onto $\mathbf{v}$ is given by

$$\mathbf{p} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{u}.$$

T/F If $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$, then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

T/F The vectors $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ form an orthonormal set of vectors in $\mathbb{R}^2$.

T/F Let $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ be an orthonormal basis for $\mathbb{R}^3$ and let $\mathbf{v} \in \mathbb{R}^3$ be a unit vector. If $\mathbf{v}^T \mathbf{u}_1 = 3/5$ and $\mathbf{v}^T \mathbf{u}_2 = 4/5$, then $\mathbf{v}$ must be orthogonal to $\mathbf{u}_3$.

T/F If $\mathbf{x}$ is an eigenvector of the non-singular matrix A, then $\mathbf{x}$ must also be an eigenvector of the inverse matrix A^{-1}.

T/F The sum $\lambda_1 + \ldots + \lambda_n$ of the eigenvalues of A must equal the determinant of A.

2. Let S be the subspace of $\mathbb{R}^4$ spanned by

$$\begin{pmatrix} 1 \\ -1 \\ 2 \\ -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2 \\ -2 \\ 5 \\ -3 \end{pmatrix}.$$

a) Write down a basis for $S^\perp$.

b) Find the distance from $\mathbf{x} = \begin{pmatrix} 2 \\ 0 \\ 3 \\ -5 \end{pmatrix}$ to $S^\perp$.

3. Find all least squares solutions to the inconsistent system of equations

$$\begin{aligned} x_1 - x_2 &= 0, \\ x_2 - x_3 &= 1, \\ x_3 - x_1 &= 2. \end{aligned}$$

4. Let

$$A = \begin{pmatrix} 1 & 2 & -4 & 1 \\ -2 & -3 & 5 & 0 \\ 3 & 4 & -6 & -1 \end{pmatrix}.$$

Find bases for $N(A)$, $R(A)$, $N(A^T)$, and $R(A^T)$.

5. Let A be an $n \times n$ matrix, B an $n \times r$ matrix, and $C = AB$. Show that

a) the null space $N(B)$ is a subspace of the null space $N(C)$,

b) the row space $R(C^T)$ of C is a subspace of the row space $R(B^T)$ of B.

[Hint: Take complements of part a), and use the fact that $N(X)^\perp = R(X^T)$ for a matrix X.]

6. Let

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 2 \end{pmatrix}.$$

a) Use the Gram-Schmidt process to find an orthonormal basis for the column space of A.

b) Using your results from part a), factor A into a product QR, where Q has orthonormal column vectors and R is upper triangular.

7. a) Find the eigenvalues and corresponding eigenvectors of $A = \begin{pmatrix} 3 & 1 \\ 5 & -1 \end{pmatrix}$.

b) Suppose that the $n \times n$ matrix A satisfies $A^2 = A$. Show that the only possible eigenvalues of A are 0 and 1.