**Name:**

Time allowed: 50 minutes.
Calculators are not allowed.

# M369 Linear Algebra (section 3) : Solutions to practise for 2nd Midterm Exam

| Problem | **T/F** | 2 | 3 | 4 | 5 | 6 | 7 | Total |
|---|---|---|---|---|---|---|---|---|
| Score | | | | | | | | |
| Maximum | 21 | 9 | 11 | 12 | 14 | 18 | 15 | 100 |

**F** If $\mathbf{x}$ is orthogonal to $\mathbf{y}$, and $\mathbf{y}$ is orthogonal to $\mathbf{z}$, then $\mathbf{x}$ must be orthogonal to $\mathbf{z}$.

*For example,* $\mathbf{x}$ *could equal* $\mathbf{z}$.

**F** If $A$ is an $m \times n$ matrix with $m > n$ then $N(A^T) = N(A)^\perp$.

$N(A^T)$ *is a subspace of* $\mathbb{R}^m$; $N(A)$ *and* $N(A)^\perp$ *are subspaces of* $\mathbb{R}^n$.

**T** If $S$ is a subspace of $\mathbb{R}^m$, and $\mathbf{b} \in S^\perp$, then $\mathbf{0}$ is the unique element of $S$ which is closest to $\mathbf{b}$.

*Since* $\mathbf{b}$ *is orthogonal to* $S$, *the projection onto* $S$ *is the zero vector* $\mathbf{0}$.

**T** Let $A$ be an $n \times n$ matrix. If the system $A\mathbf{x} = \mathbf{b}$ does not have an actual solution, then it must have infinitely many least squares solutions.

*If the system does not have an actual solution, it means that* $A$ *must be singular. Therefore the null space* $N(A) \neq \{\mathbf{0}\}$, *and adding a vector in* $N(A)$ *to any least squares solution will produce another least squares solution.*

**F** Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be the columns of an $n \times n$ orthogonal matrix $A$, with $n \geq 2$. Then

$$\|\mathbf{a}_1 + \ldots + \mathbf{a}_n\| = n.$$

$\|\mathbf{a}_1 + \ldots + \mathbf{a}_n\|^2 = \sum_{i,j=1}^n \langle \mathbf{a}_i, \mathbf{a}_j \rangle = \sum_{i,j=1}^n \delta_{ij} = n.$

**T** If $A$ is an $n \times n$ matrix which is singular, then 0 is an eigenvalue of $A$.

*If* $A$ *is singular, there exists a non-zero solution* $\mathbf{x}$ *to* $A\mathbf{x}$. *This* $\mathbf{x}$ *is a* 0*-eigenvector.*

**F** If $\mathbf{x}_1$ and $\mathbf{x}_2$ are two eigenvectors of $A$, then $\mathbf{x}_1 + \mathbf{x}_2$ must also be an eigenvector.

*If* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *correspond to different eigenvalues* $\lambda_1 \neq \lambda_2$, *then in general their sum* $\mathbf{x}_1 + \mathbf{x}_2$ *will not be an eigenvector.*

2. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ be an orthonormal basis for an inner product space $V$ and let

$$\mathbf{v} = \mathbf{u}_1 - 3\mathbf{u}_2 + 5\mathbf{u}_3 - \mathbf{u}_4 \qquad \text{and} \qquad \mathbf{w} = -3\mathbf{u}_1 + 2\mathbf{u}_2 + 6\mathbf{u}_3.$$

a) Find $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$.

$$\|\mathbf{v}\| = \sqrt{1^2 + (-3)^2 + 5^2 + (-1)^2} = \sqrt{36} = 6$$
$$\|\mathbf{w}\| = \sqrt{(-3)^2 + 2^2 + 6^2 + 0^2} = \sqrt{49} = 7$$

b) Find $\langle \mathbf{v}, \mathbf{w} \rangle$.

$$\langle \mathbf{v}, \mathbf{w} \rangle = 1.(-3) + (-3).2 + 5.6 + (-1).0 = 21$$

c) Find the angle $\theta$ between $\mathbf{v}$ and $\mathbf{w}$.

$$\cos\theta = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{21}{42} = \frac{1}{2}$$

*and therefore* $\theta = \pi/3$.

3. Let $A$ be a $9 \times 6$ matrix of rank 4.

a) Which of $R(A)$, $N(A)$, $R(A^T)$, and $N(A^T)$ are orthogonal complements in $\mathbb{R}^6$? Which are orthogonal complements in $\mathbb{R}^9$?

*The null space $N(A)$ and the row space $R(A^T)$ of $A$ are orthogonal complements in $\mathbb{R}^6$. The null space $N(A^T)$ and the column space $R(A)$ of $A$ are orthogonal complements in $\mathbb{R}^9$.*

b) What are the dimensions of the four subspaces in part a)?

*The rank is the dimension of the row space, which is also equal to the dimension of the column space. Therefore $R(A^T)$ and $R(A)$ are both four-dimensional. By the Rank-Nullity Theorem for A, the dimension of $N(A)$ (i.e. the nullity of A) is $6-4=2$. By the Rank-Nullity Theorem for $A^T$, the dimension of $N(A^T)$ (i.e. the nullity of $A^T$) is $9-4=5$.*

c) If $\mathbf{x}$ is in the row space of $A$ and $A\mathbf{x} = \mathbf{0}$, what can you say about $\|\mathbf{x}\|$? Explain your answer.

*The vector $\mathbf{x}$ is in the row space $R(A^T)$ of $A$ and in the null space $N(A)$. Since $R(A^T)$ and $N(A)$ are orthogonal complements, $\mathbf{x}$ must be the zero vector, and hence $\|\mathbf{x}\| = 0$.*

4. Let $C[0,3]$ be the space of continuous functions on the interval $[0,3]$ with inner product

$$\langle f(x), g(x)\rangle = \int_0^3 f(x)g(x)dx.$$

a) Find a linear function $ax+b$ which is orthogonal to the function $x$ in $C[0,3]$.

*We compute*

$$\langle ax+b, x\rangle = \int_0^3 (ax^2+bx)dx = 9a + \frac{9b}{2}.$$

*This will vanish provided* $b=-2a$, *for example,* $ax+b=x-2$.

b) How could you use your answer to part a) to find the best approximation to $e^x$ on $[0,3]$ by a linear function? [Write down the formula you would use but do *not* calculate the answer, i.e. do *not* calculate the integrals involved.]

*The best approximation will be given by projecting* $e^x$ *onto the subspace spanned by* $x$ *and* $x-2$. *This gives*

$$\frac{\langle e^x, x\rangle}{\langle x, x\rangle}x + \frac{\langle e^x, x-2\rangle}{\langle x-2, x-2\rangle}(x-2).$$

*Note that we need to divide by the lengths squared of* $x$ *and* $x-2$ *because we didn't rescale them to be unit vectors.*

5. Find the line $y = ax + b$ which best fits (in the sense of least squares) the data $(x, y) =$

$$(-1,1), \qquad (0,2), \qquad (1,1), \qquad \text{and} \qquad (2,3).$$

*We have to find the least squares solution to*

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}.$$

*The normal equations ($A^T A\mathbf{x} = A^T\mathbf{b}$) therefore look like*

$$\begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$$

*and the solution is*

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 7 \end{pmatrix} = \frac{1}{20} \begin{pmatrix} 4 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 3/2 \end{pmatrix}.$$

*The required line is therefore $y = x/2 + 3/2$.*

6. Let

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \left[ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \end{pmatrix} \right]$$

be a basis for $\mathbb{R}^3$.

a) Apply the Gram-Schmidt orthogonalization process to $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ to produce an orthonormal basis $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ for $\mathbb{R}^3$.

*For the first step, we normalize* $\mathbf{v}_1$*:*

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}.$$

*For the second step we project* $\mathbf{v}_2$ *onto* $\mathbf{u}_1$*, getting*

$$\mathbf{p}_1 = \langle \mathbf{v}_2, \mathbf{u}_1 \rangle \mathbf{u}_1 = \sqrt{2}\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

*Next*

$$\mathbf{v}_2 - \mathbf{p}_1 = \begin{pmatrix} 1 \\ -1 \\ \sqrt{2} \end{pmatrix}$$

*and*

$$\mathbf{u}_2 = \frac{\mathbf{v}_2 - \mathbf{p}_1}{\|\mathbf{v}_2 - \mathbf{p}_1\|} = \begin{pmatrix} 1/2 \\ -1/2 \\ \sqrt{2}/2 \end{pmatrix}.$$

*For the third step we project* $\mathbf{v}_3$ *onto* $\operatorname{Span}(\mathbf{u}_1, \mathbf{u}_2)$*, getting*

$$\mathbf{p}_2 = \langle \mathbf{v}_3, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{v}_3, \mathbf{u}_2 \rangle \mathbf{u}_2 = 0.\mathbf{u}_1 + 1.\mathbf{u}_2 = \begin{pmatrix} 1/2 \\ -1/2 \\ \sqrt{2}/2 \end{pmatrix}.$$

*Next*

$$\mathbf{v}_3 - \mathbf{p}_2 = \begin{pmatrix} -1/2 \\ 1/2 \\ \sqrt{2}/2 \end{pmatrix}$$

*and this is already a unit vector, so* $\mathbf{u}_3 = \mathbf{v}_3 - \mathbf{p}_2$ *is the vector above.*

b) Write $\mathbf{x} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{pmatrix}$ in terms of the basis $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$.

$$\begin{aligned} \mathbf{x} &= \langle \mathbf{x}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{x}, \mathbf{u}_2 \rangle \mathbf{u}_2 + \langle \mathbf{x}, \mathbf{u}_3 \rangle \mathbf{u}_3 \\ &= 2.\mathbf{u}_1 + 1.\mathbf{u}_2 + 1.\mathbf{u}_3 \end{aligned}$$

7. Let

$$A = \begin{pmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

a) Find the eigenvalues of $A$. Find at least one eigenvector of $A$.

*The characteristic polynomial is*

$$\begin{aligned} \det \begin{pmatrix} 4-\lambda & -5 & 1 \\ 1 & -\lambda & -1 \\ 0 & 1 & -1-\lambda \end{pmatrix} &= (4-\lambda)\,(-\lambda(-1-\lambda)-(-1).1) - 1.\,(-5(-1-\lambda)-1.1)+0 \\ &= -\lambda(\lambda-1)(\lambda-2) \end{aligned}$$

*and thus the eigenvalues are* $\lambda_1 = 0$, $\lambda_2 = 1$, *and* $\lambda_3 = 2$.

*The* 0*-eigenspace is just the null space of* $A$. *Solving as usual we find* $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ *(or any non-zero multiple of this vector).*

*The* 1*-eigenspace is the null space of* $A - 1.\mathrm{Id} = \begin{pmatrix} 3 & -5 & 1 \\ 1 & -1 & -1 \\ 0 & 1 & -2 \end{pmatrix}$. *Solving as usual we find* $\mathbf{x}_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ *(or any non-zero multiple of this vector). Similarly,* $\mathbf{x}_3 = \begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix}$ *is a basis for the* 2*-eigenspace.*

b) Is $A$ diagonalizable? Explain your answer.

*Since* $A$ *has three linearly independent eigenvectors, corresponding to three distinct eigenvalues,* $A$ *will be diagonalizable. Indeed, if we define*

$$X = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \begin{pmatrix} 1 & 3 & 7 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

*then* $X$ *is invertible and*

$$X^{-1}AX = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$