Name: Time allowed: 50 minutes.
Calculators are not allowed.

M369 Linear Algebra (section 3) : Practise for 2nd Midterm Exam

Problem	**T/F**	2	3	4	5	6	7	Total
Score								
Maximum	21	9	11	12	14	18	15	100

T/F If $\mathbf{x}$ is orthogonal to $\mathbf{y}$, and $\mathbf{y}$ is orthogonal to $\mathbf{z}$, then $\mathbf{x}$ must be orthogonal to $\mathbf{z}$.

T/F If A is an $m \times n$ matrix with $m > n$ then $N(A^T) = N(A)^\perp$.

T/F If S is a subspace of $\mathbb{R}^m$, and $\mathbf{b} \in S^\perp$, then $\mathbf{0}$ is the unique element of S which is closest to $\mathbf{b}$.

T/F Let A be an $n \times n$ matrix. If the system $A\mathbf{x} = \mathbf{b}$ does not have an actual solution, then it must have infinitely many least squares solutions.

T/F Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be the columns of an $n \times n$ orthogonal matrix A, with $n \geq 2$. Then

$$\|\mathbf{a}_1 + \ldots + \mathbf{a}_n\| = n.$$

T/F If A is an $n \times n$ matrix which is singular, then 0 is an eigenvalue of A.

T/F If $\mathbf{x}_1$ and $\mathbf{x}_2$ are two eigenvectors of A, then $\mathbf{x}_1 + \mathbf{x}_2$ must also be an eigenvector.

2. Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ be an orthonormal basis for an inner product space V and let

$$\mathbf{v} = \mathbf{u}_1 - 3\mathbf{u}_2 + 5\mathbf{u}_3 - \mathbf{u}_4 \qquad \text{and} \qquad \mathbf{w} = -3\mathbf{u}_1 + 2\mathbf{u}_2 + 6\mathbf{u}_3.$$

a) Find $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$.

b) Find $\langle \mathbf{v}, \mathbf{w} \rangle$.

c) Find the angle θ between $\mathbf{v}$ and $\mathbf{w}$.

3. Let A be a 9×6 matrix of rank 4.

a) Which of $R(A)$, $N(A)$, $R(A^T)$, and $N(A^T)$ are orthogonal complements in $\mathbb{R}^6$? Which are orthogonal complements in $\mathbb{R}^9$?

b) What are the dimensions of the four subspaces in part a)?

c) If $\mathbf{x}$ is in the row space of A and $A\mathbf{x} = \mathbf{0}$, what can you say about $\|\mathbf{x}\|$? Explain your answer.

4. Let $C[0,3]$ be the space of continuous functions on the interval $[0,3]$ with inner product

$$\langle f(x), g(x) \rangle = \int_0^3 f(x)g(x)dx.$$

a) Find a linear function $ax+b$ which is orthogonal to the function x in $C[0,3]$.

b) How could you use your answer to part a) to find the best approximation to e^x on $[0,3]$ by a linear function? [Write down the formula you would use but do *not* calculate the answer, i.e. do *not* calculate the integrals involved.]

5. Find the line $y = ax + b$ which best fits (in the sense of least squares) the data $(x, y) =$

$$(-1, 1), \qquad (0, 2), \qquad (1, 1), \qquad \text{and} \qquad (2, 3).$$

6. Let

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \end{pmatrix} \right]$$

be a basis for $\mathbb{R}^3$.

a) Apply the Gram-Schmidt orthogonalization process to $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$ to produce an orthonormal basis $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ for $\mathbb{R}^3$.

b) Write $\mathbf{x} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{pmatrix}$ in terms of the basis $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$.

7. Let

$$A = \begin{pmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

a) Find the eigenvalues of A. Find at least one eigenvector of A.

b) Is A diagonalizable? Explain your answer.