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Time allowed: 50 minutes.
Calculators are not allowed.

M369 Linear Algebra (section 1) : Solutions to practise for 1st Midterm Exam

1. Solve the system of equations

$$\begin{aligned} -3x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 - 3x_3 &= 0 \\ x_1 - 3x_2 + 2x_3 &= 0 \end{aligned}$$

and write down a basis for the solution space.

The coefficient matrix and reduced row echelon form are

$$\begin{pmatrix} -3 & 2 & 1 \\ 2 & 1 & -3 \\ 1 & -3 & 2 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

So x_3 *is the free variable and the solution space is*

$$\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} | x_1 = x_3, x_2 = x_3\} \quad \textit{which is the same as} \quad \{\begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\}.$$

Thus a basis for the solution space is $\left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right]$.

2. Let A be the 3×3 matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{pmatrix}.$$

a) Write down the reduced row echelon form U of A.

$$U = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

b) Use U to write down a basis for the row space of A.

A and U have the same row space, and a basis is $\{(1, 0, -1), (0, 1, 2)\}$.

c) Use U to write down a basis for the column space of A.

The column spaces of A and U are different. However, a basis for the column space of U is *given by its first two columns. Therefore the first two columns of A also give a basis for the column space of A. Thus a basis is* $\left[\begin{pmatrix}1\\1\\1\end{pmatrix},\begin{pmatrix}1\\2\\3\end{pmatrix}\right]$.

3. Determine whether the following subsets of vectors in R^2 are vector subspaces.

a) all vectors $\begin{pmatrix}x_1\\x_2\end{pmatrix}$ such that $x_1+x_2=0$;

Yes. The condition $x_1+x_2=0$ *is linear.*

b) all vectors $\begin{pmatrix}x_1\\x_2\end{pmatrix}$ such that $x_1-x_2=0$;

Yes. Once again the condition $x_1-x_2=0$ *is linear.*

c) all vectors $\begin{pmatrix}x_1\\x_2\end{pmatrix}$ such that $x_1^2-x_2^2=0$.

No. The condition $x_1^2-x_2^2$ *is quadratic, not linear. For example, the vectors* $\begin{pmatrix}1\\1\end{pmatrix}$ *and* $\begin{pmatrix}1\\-1\end{pmatrix}$ *lie in the subset, but their sum* $\begin{pmatrix}2\\0\end{pmatrix}$ *does not.*

4. Determine which of the following are linear transformations.

a) $L:\mathbb{R}^2\to\mathbb{R}^1$ defined by $L(\mathbf{x})=x_1x_2$;

Not a linear transformation. For example $L\begin{pmatrix}2\\2\end{pmatrix}=4\neq 2=2L\begin{pmatrix}1\\1\end{pmatrix}$.

b) $L:\mathbb{R}^3\to\mathbb{R}^2$ defined by $L(\mathbf{x})=\begin{pmatrix}x_1+2x_2+3x_3\\4x_1+5x_2+6\end{pmatrix}$.

Not a linear transformation. Notice that the last term of the second component is just 6, *not* $6x_3$. *Thus* $L\begin{pmatrix}2\\0\\0\end{pmatrix}=\begin{pmatrix}2\\14\end{pmatrix}\neq\begin{pmatrix}2\\20\end{pmatrix}=2L\begin{pmatrix}1\\0\\0\end{pmatrix}$.

5. Write down the transition matrix for the change of basis from $[2x-1,1-3x]$ to $[2,1-x]$ in P_2.

The transition matrix from $[2x-1,1-3x]$ *to the standard basis* $[x,1]$ *is* $V=\begin{pmatrix}2&-3\\-1&1\end{pmatrix}$ *and the transition matrix from* $[2,1-x]$ *to the standard basis* $[x,1]$ *is* $U=\begin{pmatrix}0&-1\\2&1\end{pmatrix}$. *Therefore* *the required transition matrix is*

$$S = U^{-1}V = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -2 \\ -4 & 6 \end{pmatrix}.$$

6. For each of the following sets of vectors determine whether they are linearly independent, whether they span, and whether they form a basis for the given vector space.

a) $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \left[\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}\right]$ in $\mathbb{R}^3$;

The determinant of the corresponding 3×3 *matrix is* -2. *Since this is non-zero, the three vectors are linearly independent, span* $\mathbb{R}^3$, *and hence are a basis.*

b) $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \left[\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}\right]$ in $\mathbb{R}^4$;

Three vector in $\mathbb{R}^4$ *cannot span; they cannot be a basis either. They are linearly independent.*

c) $[\mathbf{v}_1, \mathbf{v}_2] = \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]$ in $\mathbb{R}^2$.

Any set of vectors which contains the zero vector must be linearly dependent. They span the subspace $\{x_1 = x_2\}$, *not all of* $\mathbb{R}^2$. *They do not form a basis.*

7. Let A be an $m \times n$ matrix whose columns are linearly dependent. Suppose that $\mathbf{b}$ belongs to the column space of A. Prove that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

Since the columns are linearly dependent, there is a non-trivial linear combination

$$y_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + \ldots + y_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

i.e. a non-trivial solution $\mathbf{y}$ *to* $A\mathbf{y} = \mathbf{0}$. *Since* $\mathbf{b}$ *is in the column space there is a linear combination*

$$z_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + \ldots + z_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = \mathbf{b},$$

i.e. a solution $\mathbf{z}$ *to* $A\mathbf{z} = \mathbf{b}$. *Therefore for every real number* λ,

$$A(\mathbf{z} + \lambda\mathbf{y}) = A\mathbf{z} + \lambda A\mathbf{y} = \mathbf{b} + \lambda.\mathbf{0} = \mathbf{b},$$

i.e. $\mathbf{x} = \mathbf{z} + \lambda \mathbf{y}$ *is a solution to* $A\mathbf{x} = \mathbf{b}$*, and this gives infinitely many solutions.*

8. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$L(\mathbf{x}) = x_2\mathbf{b}_1 + (x_1 + x_2)\mathbf{b}_2 + x_1\mathbf{b}_3$$

where $F = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$ is the basis

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{b}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

of $\mathbb{R}^3$.

a) Find the matrix A which represents L with respect to the basis $E = [\mathbf{u}_1, \mathbf{u}_2] = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right]$ of $\mathbb{R}^2$ and the basis F of $\mathbb{R}^3$.

We calculate the columns of A:

$$\mathbf{a}_1 = [L(\mathbf{u}_1)]_F = [L\begin{pmatrix} 1 \\ 1 \end{pmatrix}]_F = \mathbf{b}_1 + 2\mathbf{b}_2 + \mathbf{b}_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad \textit{wrt the basis } F,$$

$$\mathbf{a}_2 = [L(\mathbf{u}_2)]_F = [L\begin{pmatrix} 2 \\ 1 \end{pmatrix}]_F = \mathbf{b}_1 + 3\mathbf{b}_2 + 2\mathbf{b}_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \qquad \textit{wrt the basis } F.$$

Therefore

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 2 \end{pmatrix}.$$

b) Using the matrix A, find the coordinates of $L(\mathbf{u}_1 + \mathbf{u}_2)$ with respect to the basis F.

The coordinates of $\mathbf{u}_1 + \mathbf{u}_2$ *wrt the basis* E *are* $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, *and*

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}.$$

This gives the coordinates of $L(\mathbf{u}_1 + \mathbf{u}_2)$ *with respect to the basis* F*, i.e.*

$$L(\mathbf{u}_1 + \mathbf{u}_2) = 2\mathbf{b}_1 + 5\mathbf{b}_2 + 3\mathbf{b}_3.$$