Name: Time allowed: 120 minutes.
Calculators are not allowed.

M369 Linear Algebra (section 1) : Practise Final Exam

Problem	**T/F**	2	3	4	5	6	7	8	9	Total
Score										
Maximum	?	?	?	?	?	?	?	?	?	200

T/F The set of all polynomials of degree *greater* than n (plus the zero polynomial) is a vector space.

T/F Let P_n be the vector space of polynomials of degree less than n, and let P_* be the subset of polynomials vanishing at *all* natural numbers 1, 2, 3, etc. Then P_* is a vector subspace of P_n.

T/F Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be n vectors in the vector space V. If

$$c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

implies $c_n = 0$, then $\mathbf{v}_n \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$.

T/F The three points $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ all lie in the plane $x + y + z = 2$. Therefore the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are linearly dependent.

T/F If $\mathbf{v}_1, \ldots, \mathbf{v}_m$ form a spanning set for $\mathbb{R}^n$, then the matrix $A = (\mathbf{v}_1, \ldots, \mathbf{v}_m)$ with columns $\mathbf{v}_1, \ldots, \mathbf{v}_m$ must have rank n.

T/F Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be three non-zero vectors in $\mathbb{R}^3$. If

$$\mathrm{Span}(\mathbf{v}_1) \neq \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2) \neq \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$$

then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

T/F Let $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ be a basis for $\mathbb{R}^3$, and let $U = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ be the matrix with columns $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. Then U^T is the transition matrix for the change of basis *from* the standard basis $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ *to* $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$.

T/F Let A be an $m \times n$ matrix of rank n with $m > n$. Then $N(A^T) = \{\mathbf{0}\}$.

T/F Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation given by the $m \times n$ matrix A, and $L^T : \mathbb{R}^m \to \mathbb{R}^n$ be the linear transformation given by the transpose A^T. Then the kernel of L is equal to the range of L^T.

T/F If $L : V \to W$ is a linear transformation and $\mathbf{x}$ is in the kernel of L, then $L(\mathbf{v} + \mathbf{x}) = L(\mathbf{v} - \mathbf{x})$ for all $\mathbf{v} \in V$.

T/F If A and B are similar matrices then they must have the same eigenvectors.

T/F If A_1 is similar to B_1 and A_2 is similar to B_2, then $A_1 + A_2$ must be similar to $B_1 + B_2$.

T/F The angle between two unit vectors $\mathbf{u}$ and $\mathbf{v} \in \mathbb{R}^3$ is $\cos^{-1}(\mathbf{u}^T\mathbf{v})$.

T/F The xy-plane and the yz-plane are orthogonal subspaces in $\mathbb{R}^3$.

T/F If a vector space W is the direct sum $U \oplus V$ of a pair of its subspaces U and V, then U and V must be orthogonal complements.

T/F Let A be an $m \times n$ matrix. If A^TA and AA^T are both non-singular then $m = n$.

T/F Let $\mathbb{R}^{n\times n}$ denote the vector space of $n \times n$ matrices. Then $\langle A, B\rangle = \mathrm{Tr}(AB)$ defines an inner product on $\mathbb{R}^{n\times n}$.

T/F Let $p_0(x), p_1(x), \ldots$ be a sequence of orthogonal polynomials with $p_k(x)$ of degree k for all $k \geq 0$. Then $p_n(x)$ is orthogonal to *every* polynomial of degree less than n.

T/F If $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent eigenvectors of the matrix A, then they must correspond to different eigenvalues.

T/F Let A and B be real $n \times n$ matrices. If $U = A + iB$ is unitary, then A must be orthogonal.

T/F Let $U\Sigma V^T$ be a singular value decomposition of the $m \times n$ matrix A, with non-zero singular values $\sigma_1, \ldots, \sigma_r$. Then the first r columns of V span $R(A^T)$ and the last $n-r$ columns of V span $N(A)$.

T/F If all the entries of the $n \times n$ matrix A are positive, then A must be positive definite.

2. Let $C^\infty[0,1]$ be the vector space of functions on $[0,1]$ which have derivatives of all orders. Let $S \subset C^\infty[0,1]$ be the subset of functions f satisfying

$$f''(x) + xf'(x) + (x^2 - 1)f(x) = 0.$$

Is S a vector subspace? Prove or disprove.

3. Let A be the 3×4 matrix

$$\begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 0 & 1 & 5 \\ 0 & 4 & 2 & 0 \end{pmatrix}.$$

Write down bases for $N(A)$, $R(A^T)$, $N(A^T)$, and $R(A)$.

4. Find all least squares solutions of

$$\begin{aligned} 2x_1 + x_2 &= -4 \\ x_1 - x_2 &= 1 \\ x_1 + 2x_2 &= 4. \end{aligned}$$

5. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ be two vectors in $\mathbb{R}^2$.

a) Apply the Gram-Schmidt process to $[\mathbf{v}_1, \mathbf{v}_2]$ to find an orthonormal basis $[\mathbf{u}_1, \mathbf{u}_2]$ for $\mathbb{R}^2$.

b) Find the coordinates of $\mathbf{x} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ with respect to the basis $[\mathbf{u}_1, \mathbf{u}_2]$ of part a).

6. Let

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}.$$

a) Find the eigenvalues and corresponding eigenvectors of A.

b) Use your result from part a) to write down a matrix X which diagonalizes A, and then calculate e^A.

c) Use part b) to solve the initial value problem

$$\begin{aligned} y_1' &= 2y_1 - y_2 \\ y_2' &= 3y_1 - 2y_2 \end{aligned}$$

with $y_1(0) = 2$ and $y_2(0) = 0$.

7. Let A and B be $n \times n$ matrices. Suppose that A and B are both diagonalizable, with the same diagonalizing matrix X. Prove that $AB = BA$.

8. Let

$$A = \begin{pmatrix} 1 & 7 \\ 0 & 0 \\ 1 & 7 \end{pmatrix}.$$

Find a singular value decomposition of A, i.e. factorize A into $U\Sigma V^T$ where U is a 3×3 orthogonal matrix, V is a 2×2 orthogonal matrix, and Σ looks like

$$\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq 0$.

9. Consider the symmetric matrix

$$A = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}.$$

a) Find the eigenvalues of A. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are *unit* eigenvectors corresponding to the two eigenvalues, find $|\langle \mathbf{x}_1, \mathbf{x}_2 \rangle|$.

b) Is A positive definite? Explain your answer.

c) Find the Cholesky decomposition of A, i.e. find a lower triangular matrix B with positive diagonal entries such that $A = BB^T$.

For fun and masochists (there won't be a problem on conic sections in the final exam).

T/F The conic section given by the equation $x^2 + xy + y^2 - 4 = 0$ can be put into standard position by translating in *both* the horizontal and the vertical directions.

10. Put the conic section given by the equation

$$2xy + 4\sqrt{2}x - 4\sqrt{2}y - 15 = 0$$

into standard form, by rotating and/or translating, and sketch the graph.

11. Put the conic section given by the equation

$$3x^2 + 2xy + 3y^2 + 2\sqrt{2}x + 2\sqrt{2}y - 2 = 0$$

into standard form, by rotating and/or translating, and sketch the graph.