Name: Time allowed: 120 minutes.
Calculators are not allowed.

M369 Linear Algebra (section 1) : Solutions to Practise Final Exam

Problem	**T/F**	2	3	4	5	6	7	8	9	Total
Score										
Maximum	?	?	?	?	?	?	?	?	?	200

F The set of all polynomials of degree *greater* than n (plus the zero polynomial) is a vector space.

Not closed under addition. For example, suppose $n = 2$, $p(x) = x^3 + x$, *and* $q(x) = -x^3$. *Then* $p(x) + q(x) = x$ *has degree less than two.*

T Let P_n be the vector space of polynomials of degree less than n, and let P_* be the subset of polynomials vanishing at *all* natural numbers 1, 2, 3, etc. Then P_* is a vector subspace of P_n.

In fact, a polynomial which vanishes at all natural numbers must vanish identically. So $P_* = \{0\}$ *is the trivial vector space.*

F Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be n vectors in the vector space V. If

$$c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$$

implies $c_n = 0$, then $\mathbf{v}_n \in \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$.

On the contrary, if $\mathbf{v}_n \in \textit{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$ *then there would be an equation*

$$\mathbf{v}_n = c_1\mathbf{v}_1 + \ldots + c_{n-1}\mathbf{v}_{n-1}$$

i.e. with $c_n = -1$.

F The three points $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ all lie in the plane $x+y+z = 2$. Therefore the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are linearly dependent.

The end points of the vectors lie in the plane $x+y+z = 2$, *but this plane does not pass through the origin, so the vectors themselves do not lie in this plane. One can easily check that they are independent.*

T If $\mathbf{v}_1, \ldots, \mathbf{v}_m$ form a spanning set for $\mathbb{R}^n$, then the matrix $A = (\mathbf{v}_1, \ldots, \mathbf{v}_m)$ with columns $\mathbf{v}_1, \ldots, \mathbf{v}_m$ must have rank n.

The rank of A is the dimension of the row space of A, also equal to the dimension of the column space of A. In this case, the column space is all of $\mathbb{R}^n$.

T Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be three non-zero vectors in $\mathbb{R}^3$. If

$$\text{Span}(\mathbf{v}_1) \neq \text{Span}(\mathbf{v}_1, \mathbf{v}_2) \neq \text{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$$

then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

Firstly, $\mathbf{v}_1 \neq \mathbf{0}$. Secondly, $\mathbf{v}_2$ is not in the span of $\mathbf{v}_1$. Thirdly, $\mathbf{v}_3$ is not in the span of $\mathbf{v}_1$ and $\mathbf{v}_3$. Therefore the three vectors are independent and hence form a basis.

F Let $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ be a basis for $\mathbb{R}^3$, and let $U = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ be the matrix with columns $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. Then U^T is the transition matrix for the change of basis *from* the standard basis $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ *to* $[\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$.

The transition matrix is U^{-1}, which in general is different to U^T.

F Let A be an $m \times n$ matrix of rank n with $m > n$. Then $N(A^T) = \{\mathbf{0}\}$.

By the Rank-Nullity Theorem, the dimension of $N(A^T)$ is $m - rankA = m - n > 0$.

F Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be the linear transformation given by the $m \times n$ matrix A, and $L^T : \mathbb{R}^m \to \mathbb{R}^n$ be the linear transformation given by the transpose A^T. Then the kernel of L is equal to the range of L^T.

The kernel of L is the null space $N(A)$. The range of L^T is $R(A^T)$. These two subspaces of $\mathbb{R}^n$ are orthogonal complements.

T If $L : V \to W$ is a linear transformation and $\mathbf{x}$ is in the kernel of L, then $L(\mathbf{v} + \mathbf{x}) = L(\mathbf{v} - \mathbf{x})$ for all $\mathbf{v} \in V$.

By linearity $L(\mathbf{v} + \mathbf{x}) = L(\mathbf{v}) + L(\mathbf{x}) = L(\mathbf{v})$ since $L(\mathbf{x}) = \mathbf{0}$. Similarly $L(\mathbf{v} - \mathbf{x}) = L(\mathbf{v})$.

F If A and B are similar matrices then they must have the same eigenvectors.

If $B = S^{-1}AS$ and $\mathbf{x}$ is an eigenvector of A, then $S\mathbf{x}$ will be an eigenvector of B. They will have the same eigenvalues.

F If A_1 is similar to B_1 and A_2 is similar to B_2, then $A_1 + A_2$ must be similar to $B_1 + B_2$.

This would only be true if the similarity matrices S_1 and S_2 where the same. So any randomly picked pair of similar matrices should give a counter-example.

T The angle between two unit vectors $\mathbf{u}$ and $\mathbf{v} \in \mathbb{R}^3$ is $\cos^{-1}(\mathbf{u}^T\mathbf{v})$.

If they were not unit vectors, we would also need to divide by their lengths.

F The xy-plane and the yz-plane are orthogonal subspaces in $\mathbb{R}^3$.

The vector $\mathbf{e}_2 = (0, 1, 0)$ lies in both the xy-plane and the yz-plane, so they cannot be orthogonal.

F If a vector space W is the direct sum $U \oplus V$ of a pair of its subspaces U and V, then U and V must be orthogonal complements.

For example, let $U = Span(\mathbf{e}_1)$ and $V = Span(\mathbf{e}_1 + \mathbf{e}_2)$. Then $U \cap V = \{\mathbf{0}\}$ and together U and V span $\mathbb{R}^2$, so $\mathbb{R}^2 = U \oplus V$. But they are not orthogonal.

T Let A be an $m \times n$ matrix. If A^TA and AA^T are both non-singular then $m = n$.

If the $n \times n$ matrix A^TA is non-singular it must have rank n. If the $m \times m$ matrix AA^T is non-singular it must have rank m. But A^TA and AA^T both have the same rank as A, and therefore $m = n$.

F Let $\mathbb{R}^{n\times n}$ denote the vector space of $n \times n$ matrices. Then $\langle A, B\rangle = \mathrm{Tr}(AB)$ defines an inner product on $\mathbb{R}^{n\times n}$.

For example, according to this definition

$$\left\langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle = \mathrm{Tr} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -2,$$

but this should be positive.

T Let $p_0(x), p_1(x), \ldots$ be a sequence of orthogonal polynomials with $p_k(x)$ of degree k for all $k \geq 0$. Then $p_n(x)$ is orthogonal to *every* polynomial of degree less than n.

The polynomial $p_n(x)$ is orthogonal to $p_0(x), p_1(x), \ldots, p_{n-1}(x)$, and these span the space of all polynomials of degree less than n.

F If $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent eigenvectors of the matrix A, then they must correspond to different eigenvalues.

For example, let A be the identity matrix; then every vector is an eigenvector with eigenvalue one. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be any pair of independent vectors.

F Let A and B be real $n \times n$ matrices. If $U = A + iB$ is unitary, then A must be orthogonal.

A real unitary matrix is orthogonal, but in general the real part of a unitary matrix is not orthogonal. Any randomly chosen example should illustrate this fact.

T Let $U\Sigma V^T$ be a singular value decomposition of the $m \times n$ matrix A, with non-zero singular values $\sigma_1, \ldots, \sigma_r$. Then the first r columns of V span $R(A^T)$ and the last $n - r$ columns of V span $N(A)$.

The singular values $\sigma_{r+1} = \ldots = \sigma_n = 0$ correspond to vectors in the null space of $A^T A$, which is the same as the null space of A.

F If all the entries of the $n \times n$ matrix A are positive, then A must be positive definite.

For A to be positive definite, the eigenvalues must all be positive, and this is not guaranteed by the entries of the matrix all being positive. For example

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

has eigenvalues -1 and 3.

2. Let $C^\infty[0,1]$ be the vector space of functions on $[0,1]$ which have derivatives of all orders. Let $S \subset C^\infty[0,1]$ be the subset of functions f satisfying

$$f''(x) + xf'(x) + (x^2 - 1)f(x) = 0.$$

Is S a vector subspace? Prove or disprove.

Yes, S is a vector subspace. If f and g are in S, and a and $b \in \mathbb{R}$ are scalars, then

$$(af + bg)'' + x(af + bg)' + (x^2 - 1)(af + bg) =$$

$$a(f'' + xf' + (x^2 - 1)f) + b(g'' + xg' + (x^2 - 1)g) = a.0 + b.0 = 0.$$

This shows that S is closed under vector addition and under scalar multiplication, and hence is a subspace.

3. Let A be the 3×4 matrix

$$\begin{pmatrix} 1 & 2 & 2 & 5 \\ 1 & 0 & 1 & 5 \\ 0 & 4 & 2 & 0 \end{pmatrix}.$$

Write down bases for $N(A)$, $R(A^T)$, $N(A^T)$, and $R(A)$.

The reduced row echelon forms for A and A^T are

$$U = \begin{pmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \textit{and} \qquad U' = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore the null space of A is

$$N(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \middle| \, x_1 = -x_3 - 5x_4 \textit{ and } x_2 = -\frac{x_3}{2} \right\} = \textit{Span}\left(\begin{pmatrix} -1 \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

and the null space of A^T is

$$N(A^T) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \, x_1 = -2x_3 \textit{ and } x_2 = 2x_3 \right\} = \textit{Span}\left(\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \right).$$

The row space of A is

$$R(A^T) = \textit{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2} \\ 0 \end{pmatrix} \right)$$

and the row space of A^T is

$$R(A) = \textit{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \right).$$

4. Find all least squares solutions of

$$\begin{aligned} 2x_1 + x_2 &= -4 \\ x_1 - x_2 &= 1 \\ x_1 + 2x_2 &= 4. \end{aligned}$$

The normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$ *are*

$$\begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$$

which we can simplify to

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Therefore there is a unique solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

5. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ be two vectors in $\mathbb{R}^2$.

a) Apply the Gram-Schmidt process to $[\mathbf{v}_1, \mathbf{v}_2]$ to find an orthonormal basis $[\mathbf{u}_1, \mathbf{u}_2]$ for $\mathbb{R}^2$.

First normalize $\mathbf{v}_1$ *to get* $\mathbf{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$. *The projection of* $\mathbf{v}_2$ *onto* $\mathbf{u}_1$ *is*

$$\mathbf{p}_1 = \langle \mathbf{v}_2, \mathbf{u}_1 \rangle \mathbf{u}_1 = \frac{3}{\sqrt{5}} \mathbf{u}_1 = \begin{pmatrix} \frac{3}{5} \\ \frac{6}{5} \end{pmatrix}.$$

Therefore

$$\mathbf{v}_2 - \mathbf{p}_1 = \begin{pmatrix} \frac{2}{5} \\ -\frac{1}{5} \end{pmatrix}$$

and normalizing gives

$$\mathbf{u}_2 = \frac{\mathbf{v}_2 - \mathbf{p}_1}{\|\mathbf{v}_2 - \mathbf{p}_1\|} = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}.$$

b) Find the coordinates of $\mathbf{x} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$ with respect to the basis $[\mathbf{u}_1, \mathbf{u}_2]$ of part a).

We find

$$\begin{aligned} \mathbf{x} &= \langle \mathbf{x}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{x}, \mathbf{u}_2 \rangle \mathbf{u}_2 \\ &= -\sqrt{5}\mathbf{u}_1 + 2\sqrt{5}\mathbf{u}_2. \end{aligned}$$

6. Let

$$A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}.$$

a) Find the eigenvalues and corresponding eigenvectors of A.

The characteristic equation is

$$\det \begin{pmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{pmatrix} = (2-\lambda)(-2-\lambda) - (-1).3 = \lambda^2 - 1 = (\lambda-1)(\lambda+1).$$

Therefore the eigenvalues are $\lambda_1 = 1$ *and* $\lambda_2 = -1$. *The* 1*-eigenspace is the null space of*

$$A - \mathrm{Id} = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}$$

which is spanned by $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *and similarly the* (-1)*-eigenspace is the null space of*

$$A - \mathrm{Id} = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}$$

which is spanned by $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

b) Use your result from part a) to write down a matrix X which diagonalizes A, and then calculate e^A.

Let $X = (\mathbf{x}_1, \mathbf{x}_2)$ *be the matrix with columns* $\mathbf{x}_1$ *and* $\mathbf{x}_2$. *Then* X *diagonalizes* A *and*

$$A = X \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X^{-1}.$$

Therefore

$$e^A = X \begin{pmatrix} e^1 & 0 \\ 0 & e^{-1} \end{pmatrix} X^{-1} = \frac{1}{2} \begin{pmatrix} 3e - e^{-1} & -e + e^{-1} \\ 3e - 3e^{-1} & -e + 3e^{-1} \end{pmatrix}.$$

[Part c) is on the next page.]

c) Use part b) to solve the initial value problem

$$\begin{aligned} y_1' &= 2y_1 - y_2 \\ y_2' &= 3y_1 - 2y_2 \end{aligned}$$

with $y_1(0) = 2$ and $y_2(0) = 0$.

The solution will be given by

$$\begin{aligned} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= e^{tA} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} \\ &= Xe^{tD}X^{-1} \begin{pmatrix} 2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 3e^t - e^{-t} \\ 3e^t - 3e^{-t} \end{pmatrix}. \end{aligned}$$

7. Let A and B be $n \times n$ matrices. Suppose that A and B are both diagonalizable, with the same diagonalizing matrix X. Prove that $AB = BA$.

Since $X^{-1}AX$ and $X^{-1}BX$ are both diagonal matrices, they commute, i.e.

$$(X^{-1}AX)(X^{-1}BX) = (X^{-1}BX)(X^{-1}AX).$$

Simplifying yields

$$X^{-1}ABX = X^{-1}BAX$$

and multiplying both sides on the left by X, and on the right by X^{-1} yields

$$AB = BA.$$

8. Let

$$A = \begin{pmatrix} 1 & 7 \\ 0 & 0 \\ 1 & 7 \end{pmatrix}.$$

Find a singular value decomposition of A, i.e. factorize A into $U\Sigma V^T$ where U is a 3×3 orthogonal matrix, V is a 2×2 orthogonal matrix, and Σ looks like

$$\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{pmatrix}$$

with $\sigma_1 \geq \sigma_2 \geq 0$.

We calculate

$$A^T A = \begin{pmatrix} 2 & 14 \\ 14 & 98 \end{pmatrix}.$$

The characteristic equation is

$$\det \begin{pmatrix} 2-\lambda & 14 \\ 14 & 98-\lambda \end{pmatrix} = (2-\lambda)(98-\lambda) - 14^2 = \lambda^2 - 100\lambda = \lambda(\lambda - 100).$$

Therefore the eigenvalues are $\lambda_1 = 100$ *and* $\lambda_2 = 0$ *(largest to smallest). The corresponding eigenvectors are*

$$\mathbf{v}_1 = \begin{pmatrix} \frac{1}{5\sqrt{2}} \\ \frac{7}{5\sqrt{2}} \end{pmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{pmatrix} \frac{7}{5\sqrt{2}} \\ -\frac{1}{5\sqrt{2}} \end{pmatrix}.$$

These are the columns of the matrix V*. Note that they are orthogonal and we have normalized them so that they are orthonormal, and hence* V *is an orthogonal matrix.*

The singular values are $\sigma_1 = \sqrt{\lambda_1} = 10$ *and* $\sigma_2 = \sqrt{\lambda_2} = 0$*. These determine the matrix* Σ*.*

The first column of U *is given by*

$$\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Since σ_1 *is the only non-zero singular value, the remaining columns of* U *are given by completing* $\mathbf{u}_1$ *to an orthonormal basis of* $\mathbb{R}^3$ *in whichever way we like. For example, we could take*

$$\mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \qquad \text{and} \qquad \mathbf{u}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

This gives the singular value decomposition

$$A = U\Sigma V^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{5\sqrt{2}} & \frac{7}{5\sqrt{2}} \\ \frac{7}{5\sqrt{2}} & -\frac{1}{5\sqrt{2}} \end{pmatrix}.$$

9. Consider the symmetric matrix

$$A = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}.$$

a) Find the eigenvalues of A. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are *unit* eigenvectors corresponding to the two eigenvalues, find $|\langle \mathbf{x}_1, \mathbf{x}_2 \rangle|$.

The characteristic equation is

$$\det \begin{pmatrix} 4-\lambda & 3 \\ 3 & 4-\lambda \end{pmatrix} = (4-\lambda)^2 - 3^2 = \lambda^2 - 8\lambda + 7 = (\lambda - 1)(\lambda - 7).$$

Therefore the eigenvalues are $\lambda_1 = 1$ *and* $\lambda_2 = 7$. *Since* A *is a symmetric matrix, eigenvectors corresponding to different eigenvalues will be orthogonal; thus* $|\langle \mathbf{x}_1, \mathbf{x}_2 \rangle| = 0$ *(we do not need to calculate the eigenvectors).*

b) Is A positive definite? Explain your answer.

Both eigenvalues of A *are positive. Since* A *is a symmetric matrix, this implies that it is positive definite.*

c) Find the Cholesky decomposition of A, i.e. find a lower triangular matrix B with positive diagonal entries such that $A = BB^T$.

Only one row operation is required to put A *in upper triangular form: subtracting* $\frac{3}{4}$ *times the first row from the second row. This gives*

$$\begin{pmatrix} 1 & 0 \\ -\frac{3}{4} & 1 \end{pmatrix} A = \begin{pmatrix} 4 & 3 \\ 0 & \frac{7}{4} \end{pmatrix}.$$

Factoring out a diagonal matrix gives

$$\begin{aligned} A &= \begin{pmatrix} 1 & 0 \\ \frac{3}{4} & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 0 & \frac{7}{4} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \frac{3}{4} & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & \frac{7}{4} \end{pmatrix} \begin{pmatrix} 1 & \frac{3}{4} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \frac{3}{4} & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & \frac{\sqrt{7}}{2} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & \frac{\sqrt{7}}{2} \end{pmatrix} \begin{pmatrix} 1 & \frac{3}{4} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ \frac{3}{2} & \frac{\sqrt{7}}{2} \end{pmatrix} \begin{pmatrix} 2 & \frac{3}{2} \\ 0 & \frac{\sqrt{7}}{2} \end{pmatrix}. \end{aligned}$$

Thus

$$B = \begin{pmatrix} 2 & 0 \\ \frac{3}{2} & \frac{\sqrt{7}}{2} \end{pmatrix}$$

is the required lower triangular matrix.

You're on your own for these:

T/F The conic section given by the equation $x^2 + xy + y^2 - 4 = 0$ can be put into standard position by translating in *both* the horizontal and the vertical directions.

10. Put the conic section given by the equation

$$2xy + 4\sqrt{2}x - 4\sqrt{2}y - 15 = 0$$

into standard form, by rotating and/or translating, and sketch the graph.

11. Put the conic section given by the equation

$$3x^2 + 2xy + 3y^2 + 2\sqrt{2}x + 2\sqrt{2}y - 2 = 0$$

into standard form, by rotating and/or translating, and sketch the graph.