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Time allowed: 120 minutes.
Calculators are not allowed.

M369 Linear Algebra (section 3) : Practise for the Final Exam

Problem	**T/F**	2	3	4	5	6	7	8	9	Extra Credit	Total
Score											
Maximum	45	15	18	20	18	17	22	25	20	10	200

F The set of all functions $f : [0,1] \to \mathbb{R}$ with $f''(x) = x$ is a vector space.

If f_1 and f_2 are two such functions then

$$(f_1 + f_2)'' = f_1'' + f_2'' = x + x = 2x \neq x$$

so the set is not closed under vector addition.

T Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be n vectors in the vector space V. If

$$\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}, \mathbf{v}_n) = \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$$

then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent.

If the two spans above are the same, then $\mathbf{v}_n$ must be in the span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$. This means that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent.

T If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent but do *not* span V, then V cannot have dimension n.

If V has dimension n then any set of n linearly independent vectors will be a basis, and will therefore span V.

T Let $A_1, A_2, \ldots, A_k$ be $m \times n$ matrices which span the vector space of all $m \times n$ matrices. Then k must be at least mn.

The vector space of all $m \times n$ matrices has dimension mn. Therefore a spanning set must contain at least mn vectors.

F Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Then the range of L is the set of all vectors such that $x_1 = 0$.

The range is the set of all vectors which look like $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 + x_3 \\ x_3 \end{pmatrix}$*, and this is all of* $\mathbb{R}^2$.

T If A and B are similar matrices then they must have the same eigenvalues.

The characteristic equation of B *is the same as the characteristic equation of* A*:*

$$\det(B - \lambda I) = \det(SAS^{-1} - \lambda I) = \det(S(A - \lambda I)S^{-1}) = \det(A - \lambda I)$$

F Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^3$. If $\|\mathbf{x}\| = 6$ and $\mathbf{x}^T\mathbf{y} = 3$ then $\|\mathbf{y}\|$ must be at least 2.

The Cauchy-Schwarz inequality is

$$|\mathbf{x}^T\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$$

which gives $3 \leq 6\|\mathbf{y}\|$*, i.e.,* $1/2 \leq \|\mathbf{y}\|$.

F Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^3$, and let $\mathbf{p}$ be the projection of $\mathbf{x}$ onto $\mathbf{y}$. Then $\|\mathbf{p}\| \leq \|\mathbf{y}\|$.

We know that $\mathbf{p}$ *and* $\mathbf{x} - \mathbf{p}$ *will be orthogonal and therefore Pythagoras' Thm gives*

$$\|\mathbf{x}\|^2 = \|\mathbf{p}\|^2 + \|\mathbf{x} - \mathbf{p}\|^2 \geq \|\mathbf{p}\|^2$$

and $\|\mathbf{p}\| \leq \|\mathbf{x}\|$*. However,* $\|\mathbf{y}\|$ *could be smaller than* $\|\mathbf{p}\|$.

F A least squares solution of $A\mathbf{x} = \mathbf{b}$ minimizes $||A\mathbf{x}||$.

A least squares solution will minimize $||A\mathbf{x} - \mathbf{b}||$.

T If an $m \times n$ matrix A has rank n, then A^TA is non-singular.

Both A *and* A^TA *will have the same rank, namely* n*. Thus* A^TA *is an* $n \times n$ *matrix of rank* n*, and must be non-singular.*

F If Q is an $n \times n$ orthogonal matrix, then the column space of Q and the row space of Q are orthogonal complements in $\mathbb{R}^n$.

In fact, both the columns and the rows are orthonornal bases for $\mathbb{R}^n$*, so the column space and the row space are both all of* $\mathbb{R}^n$.

F If the 4×4 matrix A has characteristic polynomial $(\lambda + 1)(\lambda - 1)^2(\lambda - 3)$ then the 1-eigenspace of A must be two-dimensional.

The 1*-eigenspace could be one or two-dimensional.*

T Let M be a Hermitian matrix, and let $\mathbf{x}_1$ and $\mathbf{x}_2$ be eigenvectors of M. If $\langle \mathbf{x}_1, \mathbf{x_2} \rangle \neq 0$ then $\mathbf{x}_1$ and $\mathbf{x}_2$ belong to the same eigenvalue.

If $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *belonged to distinct eigenvalues then they would be orthogonal, i.e.,* $\langle \mathbf{x}_1, \mathbf{x_2} \rangle$ *would be zero.*

F Let $\sigma_1, \ldots, \sigma_n$ be the singular values of A. Then $\sigma_1 = \sigma_2 = \cdots = \sigma_r = 0$, where r is the rank of A.

In fact, $\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_n = 0$.

F A symmetric positive definite matrix can have negative eigenvalues, provided their product is positive.

All of the eigenvalues of a positive definite matrix must be positive. (Note: We didn't cover this material in Spring 2007.)

2. Let S be the subset

$$\left\{ \begin{pmatrix} a+b \\ b+c \\ a \end{pmatrix} \middle| \, a,b,c \in \mathbb{R} \text{ and } a-b+c=0 \right\}$$

of $\mathbb{R}^3$. Is S a vector subspace? Prove or disprove.

The answer is yes: S is a subspace. We can use the given equation to write $c = b - a$, and substitute this into the vector to get

$$\begin{pmatrix} a+b \\ b+c \\ a \end{pmatrix} = \begin{pmatrix} a+b \\ 2b-a \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

Thus S is the span of $(1,-1,1)^T$ and $(1,2,0)^T$, which is certainly a vector subspace.

One could also show that S is a subspace by explicitly showing that it is close under scalar multiplication and vector addition. For instance, if $\begin{pmatrix} a+b \\ b+c \\ a \end{pmatrix}$ is a vector in S, with $a-b+c=0$, then

$$\lambda \begin{pmatrix} a+b \\ b+c \\ a \end{pmatrix} = \begin{pmatrix} \lambda a + \lambda b \\ \lambda b + \lambda c \\ \lambda a \end{pmatrix}.$$

Since $\lambda a - \lambda b + \lambda c = \lambda(a-b+c) = \lambda .0 = 0$ then this vector is also in S. This proves that S is closed under scalar multiplication, and closure under vector addition is similar.

3. a) Find the transition matrix corresponding to the change of basis from

$$[\mathbf{u}_1, \mathbf{u}_2] = \left[\begin{pmatrix} -4 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \end{pmatrix}\right] \qquad \text{to} \qquad [\mathbf{v}_1, \mathbf{v}_2] = \left[\begin{pmatrix} 7 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right].$$

To change from $[\mathbf{u}_1, \mathbf{u}_2]$ *to the standard basis we use* U*, and to change from the standard basis to* $[\mathbf{v}_1, \mathbf{v}_2]$ *we use* V^{-1}*. Thus the transition matrix we want is* $S = V^{-1}U$*. Remember that we first multiply a vector by* U *on the left, and then by* V^{-1} *on the left, which explains the order. In this problem*

$$S = V^{-1}U = \begin{pmatrix} 7 & 3 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -4 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -2 & 7 \end{pmatrix} \begin{pmatrix} -4 & -3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -7 & -6 \\ 15 & 13 \end{pmatrix}.$$

b) If $\mathbf{x} = 2\mathbf{u}_1 - \mathbf{u}_2$, find the coordinates of $\mathbf{x}$ with respect to $[\mathbf{v}_1, \mathbf{v}_2]$.

Using part a), the coordinates are

$$S\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -7 & -6 \\ 15 & 13 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -8 \\ 17 \end{pmatrix}.$$

4. Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + 2x_2 \\ x_2 - x_3 \end{pmatrix}.$$

Find the matrix of L with respect to the standard basis $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ of $\mathbb{R}^3$ and the basis

$$[\mathbf{v}_1, \mathbf{v}_2] = \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right]$$

of $\mathbb{R}^2$.

The i^{th} *column* $\mathbf{a}_i$ *of the matrix* A *is given by writing* $L(\mathbf{e}_i)$ *in terms of the basis* $[\mathbf{v}_1, \mathbf{v}_2]$*. To do this, we first calculate* $L(\mathbf{e}_i)$ *with respect to the standard basis, and then use the change of basis matrix*

$$V^{-1} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

Thus

$$\mathbf{a}_1 = V^{-1} L \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

Similarly

$$\mathbf{a}_2 = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \textit{and} \quad \mathbf{a}_3 = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

and thus

$$A = \begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & -2 \end{pmatrix}.$$

5. Let $C[-1,1]$ be the space of continuous functions on the interval $[-1,1]$ with inner product

$$\langle f(x), g(x)\rangle = \frac{1}{2}\int_{-1}^{1} f(x)g(x)dx.$$

a) Verify that $\{1, x\}$ is an orthogonal set with respect to this inner product.

$$\langle 1, x\rangle = \frac{1}{2}\int_{-1}^{1} xdx = \frac{1}{4}x^2|_{-1}^{1} = \frac{1}{4}(1^2 - (-1)^2) = 0$$

so 1 *and* x *are orthogonal.*

b) Normalization $\{1, x\}$ to obtain an orthonormal set.

$$\|1\|^2 = \langle 1, 1\rangle = \frac{1}{2}\int_{-1}^{1} dx = 1$$

so 1 *is already a unit vector.*

$$\|x\|^2 = \langle x, x\rangle = \frac{1}{2}\int_{-1}^{1} x^2dx = \frac{1}{6}x^3|_{-1}^{1} = \frac{1}{6}(1^3 - (-1)^3) = \frac{1}{3}$$

so we need to divide x *by* $\frac{1}{\sqrt{3}}$*. The resulting unit vector is* $\sqrt{3}x$*.*

c) Using the orthonormal set from part b), find the best (least squares) approximation of $3x^2$ on $[-1,1]$ by a linear function $ax + b$.

The best approximation is given by projecting $3x^2$ *onto the subspace spanned by the orthonormal basis* $\{1, \sqrt{3}x\}$ *for the subspace of linear functions. The formula for this is*

$$\begin{aligned}
\langle 3x^2, 1\rangle 1 + \langle 3x^2, \sqrt{3}x\rangle\sqrt{3}x &= \left(\frac{1}{2}\int_{-1}^{1} 3x^2dx\right)1 + \left(\frac{1}{2}\int_{-1}^{1} 3\sqrt{3}x^3dx\right)\sqrt{3}x \\
&= \left(\frac{1}{2}x^3|_{-1}^{1}\right)1 + \left(\frac{3\sqrt{3}}{8}x^4|_{-1}^{1}\right)\sqrt{3}x \\
&= 1.
\end{aligned}$$

6. Find the line $y = ax + b$ which best fits (in the sense of least squares) the data $(x, y) =$

$$(0,0), \qquad (1,2), \qquad \text{and} \qquad (2,2).$$

In this problem a and b are the unknowns and the three points give three equations for a and b, which look like $xa + b = y$. Thus we obtain the overdetermined system

$$\begin{aligned} 0.a + 1.b &= 0 \\ 1.a + 1.b &= 2 \\ 2.a + 1.b &= 2. \end{aligned}$$

The corresponding normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$ look like

$$\begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}.$$

The solution is therefore

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 3 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 3 & -3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{3} \end{pmatrix}$$

and the line is $y = x + \frac{1}{3}$.

7. Consider the Hermitian matrix

$$A = \begin{pmatrix} 3 & 2+i \\ 2-i & -1 \end{pmatrix}.$$

Find a unitary matrix which diagonalizes A.

We first find the eigenvalues by solving

$$\begin{aligned} \det(A-\lambda I) &= \det\begin{pmatrix} 3-\lambda & 2+i \\ 2-i & -1-\lambda \end{pmatrix} \\ &= (3-\lambda)(-1-\lambda)-(2+i)(2-i) \\ &= \lambda^2-2\lambda-8 \\ &= (\lambda-4)(\lambda+2). \end{aligned}$$

Thus the eigenvalues are $\lambda_1 = 4$ *and* $\lambda_2 = -2$.

The eigenvector corresponding to λ_1 *can be found be solving*

$$A-\lambda_1 I = \begin{pmatrix} 3-4 & 2+i \\ 2-i & -1-4 \end{pmatrix} = \begin{pmatrix} -1 & 2+i \\ 2-i & -5 \end{pmatrix}$$

which has reduced row echelon form $\begin{pmatrix} -1 & 2+i \\ 0 & 0 \end{pmatrix}$. *Therefore the eigenvector is*

$$\mathbf{x}_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 2+i \\ 1 \end{pmatrix}.$$

Note that we have normalized to make it a unit vector.

Similarly, we find that the eigenvector corresponding to λ_2 *is* $\mathbf{x}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2-i \end{pmatrix}$. *One can check that this is orthogonal to* $\mathbf{x}_1$. *The unitary matrix* X *is therefore*

$$\frac{1}{\sqrt{6}}\begin{pmatrix} 2+i & -1 \\ 1 & 2-i \end{pmatrix}$$

and

$$X^H A X = D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}.$$

8. Let

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & 2 \end{pmatrix}.$$

a) Find the singular values of A.

We first find the eigenvalues of $A^TA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & -2 \\ 0 & -2 & 5 \end{pmatrix}$, *which are the zeros of*

$$\begin{aligned} \det(A^TA - \lambda I) &= \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 8-\lambda & -2 \\ 0 & -2 & 5-\lambda \end{pmatrix} \\ &= (1-\lambda)((8-\lambda)(5-\lambda) - (-2)^2) \\ &= (1-\lambda)(\lambda^2 - 13\lambda + 36) \\ &= -(\lambda-1)(\lambda-4)(\lambda-9). \end{aligned}$$

Therefore the eigenvalues (in order largest to smallest) are $\lambda_1 = 9$, $\lambda_2 = 4$, *and* $\lambda_3 = 1$. *The singular values are the square roots of these:* $\sigma_1 = 3$, $\sigma_2 = 2$, *and* $\sigma_3 = 1$.

b) Find a rank one matrix which best approximates A.

[Hint: You do not need to calculate the entire singular value decomposition $U\Sigma V^T$ *of* A*; it suffices to find the first columns of* U *and* V*.]*

The best approximation will be given by $\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T$. *Here* $\mathbf{v}_1$ *is a unit eigenvector of* A^TA *belonging to the eigenvalue* $\lambda_1 = 9$. *We calculate the reduced row echelon form of*

$$A^TA - \lambda_1 I = \begin{pmatrix} -8 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore a normalized eigenvector is $\mathbf{v}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$. *Then* $\mathbf{u}_1$ *is given by*

$$\frac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{3} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & 2 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \frac{1}{3\sqrt{5}} \begin{pmatrix} 0 \\ 3 \\ -6 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}.$$

Finally

$$\sigma_1 \mathbf{u}_1 \mathbf{v}_1^T = 3\frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \frac{1}{\sqrt{5}}(0, 2, -1) = \frac{3}{5} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{6}{5} & \frac{-3}{5} \\ 0 & \frac{-12}{5} & \frac{6}{5} \end{pmatrix}.$$

9. Consider the symmetric matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

a) Find the eigenvalues of A.

The eigenvalues are the zeros of

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} \\ &= (2-\lambda)(2-\lambda) - 1^2 \\ &= \lambda^2 - 4\lambda + 3 \\ &= (\lambda - 1)(\lambda - 3). \end{aligned}$$

Therefore the eigenvalues are $\lambda_1 = 1$ *and* $\lambda_2 = 3$.

b) Is A positive definite? Explain your answer.

Both eigenvalues are positive and A is symmetric. Therefore A is positive definite. (Note: We did not cover this in Spring 2007.)

c) Find the Cholesky decomposition of A, i.e. find a lower triangular matrix B with positive diagonal entries such that $A = BB^T$.

Just one row operation is required to make A upper triangular: we substract half the first row from the second row. Therefore

$$A = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & \frac{3}{2} \end{pmatrix}.$$

Next we decompose the matrix on the right into the product of a diagonal matrix D and an upper triangular matrix with ones along the diagonal:

$$\begin{pmatrix} 2 & 1 \\ 0 & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

Finally, writing $D = D^{1/2}D^{1/2}$ yields the Cholesky decomposition:

$$A = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{\frac{3}{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{\frac{3}{2}} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} & 0 \\ \frac{\sqrt{2}}{2} & \sqrt{\frac{3}{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & \sqrt{\frac{3}{2}} \end{pmatrix}$$

(Note: This material was not covered in Spring 2007.)

[Extra credit] Let A be a 3×3 matrix of rank 2 with characteristic polynomial $\lambda^2 - \lambda^3$. Can A be diagonalized? Explain your answer.

The characteristic polynomial is $\lambda^2 - \lambda^3 = \lambda^2(1 - \lambda)$. *Therefore* A *has eigenvalues* $\lambda_1 = 1$ *and* $\lambda_2 = \lambda_3 = 0$, *i.e., the eigenvalue zero has multiplicity two. Thus* A *will be diagonalizable if and only if the* 0*-eigenspace has dimension two, so that (together with an eigenvector corresponding to* $\lambda_1 = 1$*) there will be three linearly independent eigenvectors. However, by the Rank-Nullity Theorem, the dimension of the* 0*-eigenspace is*

$$\mathrm{dim} N(A) = n - \mathrm{rank} A = 3 - 2 = 1.$$

Therefore the 0*-eigenspace is only one-dimensional, and there are not enough linearly independent eigenvectors to diagonalize* A.