Name: Time allowed: 120 minutes.
Calculators are not allowed.

M369 Linear Algebra (section 3) : Practise for the Final Exam

Problem	**T/F**	2	3	4	5	6	7	8	9	Extra Credit	Total
Score											
Maximum	45	15	18	20	18	17	22	25	20	10	200

T/F The set of all functions $f : [0,1] \to \mathbb{R}$ with $f''(x) = x$ is a vector space.

T/F Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be n vectors in the vector space V. If

$$\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}, \mathbf{v}_n) = \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$$

then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent.

T/F If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent but do *not* span V, then V cannot have dimension n.

T/F Let $A_1, A_2, \ldots, A_k$ be $m \times n$ matrices which span the vector space of all $m \times n$ matrices. Then k must be at least mn.

T/F Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by the matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Then the range of L is the set of all vectors such that $x_1 = 0$.

T/F If A and B are similar matrices then they must have the same eigenvalues.

T/F Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^3$. If $\|\mathbf{x}\| = 6$ and $\mathbf{x}^T\mathbf{y} = 3$ then $\|\mathbf{y}\|$ must be at least 2.

T/F Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^3$, and let $\mathbf{p}$ be the projection of $\mathbf{x}$ onto $\mathbf{y}$. Then $\|\mathbf{p}\| \le \|\mathbf{y}\|$.

T/F A least squares solution of $A\mathbf{x} = \mathbf{b}$ minimizes $\|A\mathbf{x}\|$.

T/F If an $m \times n$ matrix A has rank n, then $A^T A$ is non-singular.

T/F If Q is an $n \times n$ orthogonal matrix, then the column space of Q and the row space of Q are orthogonal complements in $\mathbb{R}^n$.

T/F If the 4×4 matrix A has characteristic polynomial $(\lambda + 1)(\lambda - 1)^2(\lambda - 3)$ then the 1-eigenspace of A must be two-dimensional.

T/F Let M be a Hermitian matrix, and let $\mathbf{x}_1$ and $\mathbf{x}_2$ be eigenvectors of M. If $\langle \mathbf{x}_1, \mathbf{x_2} \rangle \neq 0$ then $\mathbf{x}_1$ and $\mathbf{x}_2$ belong to the same eigenvalue.

T/F Let $\sigma_1, \ldots, \sigma_n$ be the singular values of A. Then $\sigma_1 = \sigma_2 = \cdots = \sigma_r = 0$, where r is the rank of A.

T/F A symmetric positive definite matrix can have negative eigenvalues, provided their product is positive.

2. Let S be the subset

$$\left\{ \begin{pmatrix} a+b \\ b+c \\ a \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \text{ and } a - b + c = 0 \right\}$$

of $\mathbb{R}^3$. Is S a vector subspace? Prove or disprove.

3. a) Find the transition matrix corresponding to the change of basis from

$$[\mathbf{u}_1, \mathbf{u}_2] = \left[\begin{pmatrix} -4 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right] \quad \text{to} \quad [\mathbf{v}_1, \mathbf{v}_2] = \left[\begin{pmatrix} 7 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right].$$

b) If $\mathbf{x} = 2\mathbf{u}_1 - \mathbf{u}_2$, find the coordinates of $\mathbf{x}$ with respect to $[\mathbf{v}_1, \mathbf{v}_2]$.

4. Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + 2x_2 \\ x_2 - x_3 \end{pmatrix}.$$

Find the matrix of L with respect to the standard basis $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]$ of $\mathbb{R}^3$ and the basis

$$[\mathbf{v}_1, \mathbf{v}_2] = \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right]$$

of $\mathbb{R}^2$.

5. Let $C[-1,1]$ be the space of continuous functions on the interval $[-1,1]$ with inner product

$$\langle f(x), g(x)\rangle = \frac{1}{2}\int_{-1}^{1} f(x)g(x)dx.$$

a) Verify that $\{1, x\}$ is an orthogonal set with respect to this inner product.

b) Normalization $\{1, x\}$ to obtain an orthonormal set.

c) Using the orthonormal set from part b), find the best (least squares) approximation of $3x^2$ on $[-1,1]$ by a linear function $ax+b$.

6. Find the line $y = ax + b$ which best fits (in the sense of least squares) the data $(x, y) =$

$$(0,0), \qquad (1,2), \qquad \text{and} \qquad (2,2).$$

7. Consider the Hermitian matrix

$$A = \begin{pmatrix} 3 & 2+i \\ 2-i & -1 \end{pmatrix}.$$

Find a unitary matrix which diagonalizes A.

8. Let

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & 2 \end{pmatrix}.$$

a) Find the singular values of A.

b) Find a rank one matrix which best approximates A.

[Hint: You do not need to calculate the entire singular value decomposition $U\Sigma V^T$ of A; it suffices to find the first columns of U and V.]

9. Consider the symmetric matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

a) Find the eigenvalues of A.

b) Is A positive definite? Explain your answer.

c) Find the Cholesky decomposition of A, i.e. find a lower triangular matrix B with positive diagonal entries such that $A = BB^T$.

[Extra credit] Let A be a 3×3 matrix of rank 2 with characteristic polynomial $\lambda^2 - \lambda^3$. Can A be diagonalized? Explain your answer.