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Time allowed:
50 minutes.
Calculators are not allowed.

M369 Linear Algebra (section 3) : Solutions to 2nd Midterm Exam

Problem	**T/F**	2	3	4	5	6	7	Total
Score								
Maximum	21	15	12	10	15	15	12	100

F If $\langle \mathbf{v}, \mathbf{w} \rangle = 0$ then $\mathbf{v}$ and $\mathbf{w}$ must be linearly dependent.

If the inner product is zero the vectors are orthogonal; so they could only be linearly dependent if one of the vectors is zero.

F Let A be an $n \times n$ matrix. The column space and the row space of A must be orthogonal complements.

For example, let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. *The row and column spaces are both spanned by* $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, *which is certainly not orthogonal to itself.*

T An actual solution of $A\mathbf{x} = \mathbf{b}$ must also be a least squares solution.

A least squares solutions is a minimizer of $\|A\mathbf{x} - \mathbf{b}\|$. *If* $A\mathbf{x} = \mathbf{b}$ *then* $\|A\mathbf{x} - \mathbf{b}\| = 0$, *which is certainly a minimum.*

T Let S be a subspace of $\mathbb{R}^n$. If $\mathbf{p}$ is the projection of a vector $\mathbf{v} \in \mathbb{R}^n$ onto S, then $\mathbf{p}$ must be orthogonal to $\mathbf{v} - \mathbf{p}$.

The projection $\mathbf{p}$ *lies in* S, *and is uniquely defined by requiring that* $\mathbf{v} - \mathbf{p}$ *be orthogonal to* S. *In particular,* $\mathbf{p}$ *and* $\mathbf{v} - \mathbf{p}$ *must be orthogonal.*

T If Q is an $n \times n$ matrix which is orthogonal then Q must be invertible.

In fact $Q^{-1} = Q^T$, *so* Q^{-1} *exists.*

F If an $n \times n$ matrix A is diagonalizable then A must have n distinct eigenvalues.

A diagonalizable matrix can have an eigenvalue with multiplicity, provided the dimension of the corresponding eigenspace equals the multiplicity of the eigenvalue.

F If $\mathbf{x}_1$ and $\mathbf{x}_2$ are both eigenvectors of A corresponding to the eigenvalue 2, then $\mathbf{x}_1 + \mathbf{x}_2$ must be an eigenvector of A corresponding to the eigenvalue 4.

Eigenvectors corresponding to the eigenvalue 2 *form a subspace, so if we add* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *we get another eigenvector with eigenvalue* 2.

2. a) Find a basis for the subspace $S = \{\mathbf{x} \in \mathbb{R}^3 | x_1 + x_2 + x_3 = 0\}$ of $\mathbb{R}^3$.

The solution space (S) *is given by* $x_1 = -x_2 - x_3$, *thus*

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

and a basis for S *is* $\{\mathbf{v}_1, \mathbf{v_2}\} = \{(-1, 1, 0)^T, (-1, 0, 1)^T\}$.

b) Apply the Gram-Schmidt process to find an orthonormal basis for S.

First we normalize $\mathbf{v}_1$ *to get*

$$\mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}.$$

Next we project $\mathbf{v}_2$ *onto* $\mathbf{u}_1$ *to get*

$$\mathbf{p}_1 = \langle \mathbf{v}_2, \mathbf{u}_1 \rangle \mathbf{u}_1 = \frac{1}{\sqrt{2}} \mathbf{u}_1 = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix}.$$

Thus

$$\mathbf{v}_2 - \mathbf{p}_1 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix}.$$

Finally we normalize to get

$$\mathbf{u}_2 = \frac{\mathbf{v}_2 - \mathbf{p}_1}{\|\mathbf{v}_2 - \mathbf{p}_1\|} = \begin{pmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}.$$

c) Use the orthonormal basis to find the point in S which is closest to $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

The closest vector is given by projecting $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ *onto* S, *and this projection is given by*

$$\langle \mathbf{w}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \langle \mathbf{w}, \mathbf{u}_2 \rangle \mathbf{u}_2 = \frac{1}{\sqrt{2}} \mathbf{u}_1 + \frac{3}{\sqrt{6}} \mathbf{u}_2 = \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

3. a) Let $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Find $\|\mathbf{v}\|$ and $\|\mathbf{w}\|$.

$$\|\mathbf{v}\| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$$

and

$$\|\mathbf{w}\| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$$

.

b) Find $\langle \mathbf{v}, \mathbf{w} \rangle$.

$$\langle \mathbf{v}, \mathbf{w} \rangle = 1.2 + 1.0 + 2.1 = 3.$$

c) Find the angle θ between $\mathbf{v}$ and $\mathbf{w}$.

$$\theta = \cos^{-1}\left(\frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\|.\|\mathbf{w}\|}\right) = \cos^{-1}\left(\frac{3}{\sqrt{6}.\sqrt{2}}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}.$$

d) Find the projection of $\mathbf{w}$ onto $\mathbf{v}$.

The projection is

$$\mathbf{p} = \frac{\langle \mathbf{w}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v} = \frac{3}{6}\mathbf{v} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix}.$$

4. Find all least squares solutions of the system of equations

$$\begin{aligned} x_1 + 2x_2 &= 0, \\ x_1 + 2x_2 &= 1, \\ x_1 + 2x_2 &= 5. \end{aligned}$$

The normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$ *look like*

$$\begin{pmatrix} 3 & 6 \\ 6 & 12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}.$$

The reduced row echelon form for this system of equations is

$$\left(\begin{array}{cc|c} 1 & 2 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

so x_2 *is the free variable and*

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 - 2x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

5. Let A be a 7×5 matrix of rank 4 and let $\mathbf{b}$ be a non-zero vector in $N(A^T)$.

a) What are the dimensions of $R(A)$, $N(A)$, $R(A^T)$, and $N(A^T)$?

Both the column space $R(A)$ and the row space $R(A^T)$ have dimension equal to the rank of A, namely 4. By the Rank-Nullity Theorem

$$\dim N(A) = n - \operatorname{rank} A = 5 - 4 = 1$$

and

$$\dim N(A^T) = m - \operatorname{rank} A^T = 7 - 4 = 3.$$

b) How many solutions of $A\mathbf{x} = \mathbf{b}$ will there be? Explain your answer.

We are told that $\mathbf{b}$ lies in $N(A^T)$, which is the orthogonal complement to $R(A)$. The only vector which lies in $R(A)$ and $R(A)^\perp$ is the zero vector, but we are also told that $\mathbf{b}$ is non-zero. Therefore $\mathbf{b}$ does not lie in the column space $R(A)$. This means that $A\mathbf{x} = \mathbf{b}$ cannot have a solution.

c) Explain why $\hat{\mathbf{x}} = \mathbf{0}$ is a least squares solution of $A\mathbf{x} = \mathbf{b}$.

Since $\mathbf{b}$ lies in $N(A^T)$, we have $A^T\mathbf{b} = \mathbf{0}$. Therefore the normal equations are $A^T A\hat{\mathbf{x}} = A^T\mathbf{b} = \mathbf{0}$. Clearly $\hat{\mathbf{x}} = \mathbf{0}$ is a solution to the normal equations, and therefore a least squares solution of $A\mathbf{x} = \mathbf{b}$.

d) Are there other least squares solutions of $A\mathbf{x} = \mathbf{b}$? Explain your answer.

Any solution to the normal equations $A^T A\hat{\mathbf{x}} = \mathbf{0}$ is a least squares solution. For example, if $\hat{\mathbf{x}}$ lies in the null-space $N(A)$ then $A\hat{\mathbf{x}} = \mathbf{0}$ and hence $A^T A\hat{\mathbf{x}}$ is also $\mathbf{0}$. Since $N(A)$ is one-dimensional, this gives a one-dimensional space of least squares solutions (i.e., infinitely many solutions).

6. Let $C[0,1]$ be the space of continuous functions on the interval $[0,1]$ with the standard inner product

$$\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)dx.$$

a) Apply the Gram-Schmidt process to $\{1, x\}$ to find an orthonormal basis for the subspace of linear functions.

First we normalize 1. *Since* $\|1\| = \left(\int_0^1 1^2 dx\right)^{1/2} = 1$ *we find*

$$\mathbf{u}_1 = \frac{1}{\|1\|} = 1.$$

Next we project x *onto* $\mathbf{u}_1 = 1$ *to get*

$$\mathbf{p}_1 = \langle x, 1\rangle 1 = \left(\int_0^1 x dx\right) 1 = \frac{1}{2}.$$

Thus $x - \mathbf{p}_1 = x - \frac{1}{2}$. *Finally we use*

$$\|x - \frac{1}{2}\| = \left(\int_0^1 (x - \frac{1}{2})^2 dx\right)^{1/2} = \ldots = \sqrt{\frac{1}{12}}$$

to normalize $x - \frac{1}{2}$, *obtaining*

$$\mathbf{u}_2 = \frac{x - \frac{1}{2}}{\|x - \frac{1}{2}\|} = \sqrt{12}(x - \frac{1}{2}) = \sqrt{3}(2x - 1).$$

b) Use the orthonormal basis to find the best approximation to $\ln(x+1)$ on $[0,1]$ by a linear function.

[Hint: $\int_0^1 \ln(x+1)dx = 2\ln 2 - 1$ and $\int_0^1 x\ln(x+1)dx = \frac{1}{4}$]

The best approximation is given by projecting $f = \ln(x+1)$ *onto* $\mathrm{Span}(1, x) = \mathrm{Span}(\mathbf{u}_1, \mathbf{u}_2)$. *This is given by the formula*

$$\begin{aligned}
\langle f, \mathbf{u}_1\rangle\mathbf{u}_1 + \langle f, \mathbf{u}_2\rangle\mathbf{u}_2 &= \langle \ln(x+1), 1\rangle 1 + \langle \ln(x+1), \sqrt{3}(2x-1)\rangle\sqrt{3}(2x-1) \\
&= (2\ln 2 - 1)1 + (2\langle \ln(x+1), x\rangle - \langle \ln(x+1), 1\rangle)3(2x-1) \\
&= (2\ln 2 - 1)1 + (\frac{1}{2} - (2\ln 2 - 1))3(2x-1) \\
&= \ldots \\
&= (8\ln 2 - \frac{11}{2}) + (9 - 12\ln 2)x.
\end{aligned}$$

7. a) One of the eigenvalues of the matrix

$$A = \begin{pmatrix} -1 & 6 \\ -1 & 4 \end{pmatrix}$$

is $\lambda_1 = 1$, with corresponding eigenvector $\mathbf{x}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Find the second eigenvalue and corresponding eigenvector.

The characteristic equation is

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} -1-\lambda & 6 \\ -1 & 4-\lambda \end{vmatrix} \\ &= (\lambda+1)(\lambda-4)+6 \\ &= \lambda^2 - 3\lambda + 2 \\ &= (\lambda-1)(\lambda-2) \end{aligned}$$

and therefore the other eigenvalue is $\lambda_2 = 2$. *The* $2-$*eigenspace is the null space of*

$$A - 2I = \begin{pmatrix} -3 & 6 \\ -1 & 2 \end{pmatrix}$$

which has reduced row echelon form

$$\left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

Therefore $\mathbf{x}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ *(or any non-zero multiple of* $\mathbf{x}_2$*) is an eigenvector with eigenvalue* 2.

b) Find a matrix X which diagonalizes A.

We can form the matrix X *whose columns are the eigenvectors of* A*. Thus*

$$X = (\mathbf{x}_1, \mathbf{x}_2) = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

and then

$$X^{-1}AX = D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$