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Time allowed: 50 minutes.
Calculators are not allowed.

M369 Linear Algebra (section 1) : Solutions to 1st Midterm Exam

Problem	1	2	3	4	5	6	Extra Credit	Total
Score								
Maximum	12	15	18	20	15	20	(15)	100

1. Determine whether the following subsets of vectors in R^3 are vector subspaces.

a) all vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ such that $x_1 - x_2 + x_3 = 0$;

Yes, it is a subspace. If $\mathbf{x}$ *and* $\mathbf{y}$ *satisfy* $x_1 - x_2 + x_3 = 0$ *and* $y_1 - y_2 + y_3 = 0$ *respectively, then their sum* $\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{pmatrix}$ *satisfies*

$$(x_1 + y_1) - (x_2 + y_2) + (x_3 + y_3) = (x_1 - x_2 + x_3) + (y_1 - y_2 + y_3) = 0$$

and the scalar multiple $\alpha\mathbf{x} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{pmatrix}$ *satisfies*

$$\alpha x_1 - \alpha x_2 + \alpha x_3 = \alpha(x_1 - x_2 + x_3) = 0.$$

Therefore the subset is closed under vector addition and scalar multiplication. It is clearly non-empty.

b) all vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ such that $x_1 x_2 x_3 = 0$.

No, it is not a subspace. Rather than writing down a general equation here, it is better to find a simple example: $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ *both belong to the subset, but their sum* $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ *does not. Therefore the subset is not closed under vector addition. Note however that it is closed under scalar multiplication, for if* $\mathbf{x}$ *satisfies* $x_1 x_2 x_3 = 0$ *then* $\alpha\mathbf{x} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{pmatrix}$ *satisfies*

$$(\alpha x_1)(\alpha x_2)(\alpha x_3) = \alpha^3(x_1 x_2 x_3) = \alpha^3 .0 = 0.$$

2. Solve the system of equations

$$\begin{aligned} x_1 + 3x_2 + x_3 + 2x_4 &= 0 \\ 2x_1 + 7x_2 + 3x_3 + 5x_4 &= 0 \\ 3x_1 + 7x_2 + x_3 + 4x_4 &= 0 \end{aligned}$$

and write down a basis for the solution space.

The coefficient matrix and reduced row echelon form are

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 7 & 3 & 5 \\ 3 & 7 & 1 & 4 \end{pmatrix} \qquad \textit{and} \qquad \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So x_3 and x_4 are the free variables and the solution is all $\mathbf{x} \in \mathbb{R}^4$ such that $x_1 = 2x_3 + x_4$ and $x_2 = -x_3 - x_4$. In other words, the solution space is

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \middle| \begin{array}{c} x_1 = 2x_3 + x_4 \\ x_2 = -x_3 - x_4 \end{array} \right\}$$

which is the same as

$$\left\{ \begin{pmatrix} 2x_3 + x_4 \\ -x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Thus a basis for the solution space is $\left[\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right].$

3. For each of the following sets of vectors in $\mathbb{R}^3$, determine whether they are linearly independent, whether they span $\mathbb{R}^3$, and whether they form a basis for $\mathbb{R}^3$.

a) $[\mathbf{v}_1, \mathbf{v}_2] = \left[\begin{pmatrix} 4 \\ 9 \\ 1 \end{pmatrix}, \begin{pmatrix} 9 \\ 1 \\ 4 \end{pmatrix} \right]$;

Two vectors in $\mathbb{R}^3$ cannot span and cannot form a basis. They are linearly independent since neither vector is a multiple of the other.

b) $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \left[\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right]$;

The determinant of the corresponding 3×3 matrix is 9. Since this is non-zero the vectors are linearly independent. Now in general n vectors in $\mathbb{R}^n$ are linearly independent if and only if they span $\mathbb{R}^n$. Therefore the vectors also span $\mathbb{R}^3$, and thus form a basis. We could also consider the corresponding 3×3 matrix, and show that it's reduced row echelon form is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This has rank 3, so the vectors span $\mathbb{R}^3$ and are linearly independent.

c) $[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4] = \left[\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right]$.

Four vectors in $\mathbb{R}^3$ must be linearly dependent. Moreover, any set of vectors containing the zero vector must be linearly dependent. This implies that they cannot form a basis. They span as an arbitrary vector in $\mathbf{x} \in \mathbb{R}^3$ can be written

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (x_3 - x_2) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + (x_2 - x_1) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + x_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

This can also be seen by calculating the reduced row echelon form of the corresponding 3×4 matrix, and observing that it has rank 3. However, it does not make sense to talk about the determinant of this matrix: only square matrices have determinants, whereas this matrix is 3×4.

4. Let A be an $m \times n$ matrix.

a) Define the rank and the nullity of A, and state the Rank-Nullity Theorem.

The rank of A is the dimension of the row space of A. The nullity is the dimension of the null space $N(A) = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} = \mathbf{0}\}$. The Rank-Nullity Theorem says that the rank plus the nullity of A is equal to n, which is the number of columns of A.

b) Prove that if the rank of A is less than n, then there exists a non-zero vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{0}$.

Suppose the rank r of A is less than n. By the Rank-Nullity Theorem, the nullity is equal to $n - r$, which must be greater than zero since $r < n$. Therefore the null space $N(A)$ has positive dimension, and in particular is not just the zero vector, $N(A) \neq \{\mathbf{0}\}$. Thus there exists a non-zero vector $\mathbf{x} \in \mathbb{R}^n$ in the null space $N(A)$, i.e. a vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$.

Another way to solve this is to observe that the rank r is the number of non-zero rows in the reduced row echelon form U of A. Equivalently, r is the number of lead variables. Since there are n columns, this means there are $n - r > 0$ free variables. So there is at least one free variable, which means $U\mathbf{x} = \mathbf{0}$ has a non-trivial solution $\mathbf{x}$. This non-zero vector $\mathbf{x}$ will also be a solution of $A\mathbf{x} = \mathbf{0}$.

Warning: If you mention "vectors of A" you should specify whether you mean row or column vectors of A. Also, "dimension of A" doesn't really have a meaning. Similarly, the determinant of A only exists when $m = n$.

5. Determine which of the following are linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^2$.

a) $L(\mathbf{x}) = \begin{pmatrix} 0 \\ x_1 + x_2^2 + x_3^3 \end{pmatrix}$;

No, this is not a linear transformation. For example,

$$L\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} \neq 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2L\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

so L is not compatible with scalar multiplication.

b) $L(\mathbf{x}) = \begin{pmatrix} 2x_2 \\ (x_1+1)^2 - (x_1-1)^2 \end{pmatrix}$.

[Hint: First expand and simplify the right hand side.]

Yes, this is a linear transformation. Expanding and simplifying gives

$$L(\mathbf{x}) = \begin{pmatrix} 2x_2 \\ (x_1^2 + 2x_1 + 1) - (x_1^2 - 2x_1 + 1) \end{pmatrix} = \begin{pmatrix} 2x_2 \\ 4x_1 \end{pmatrix}.$$

Therefore

$$L(\alpha\mathbf{x}) = L\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{pmatrix} = \begin{pmatrix} 2\alpha x_2 \\ 4\alpha x_1 \end{pmatrix} = \alpha\begin{pmatrix} 2x_2 \\ 4x_1 \end{pmatrix} = \alpha L(\mathbf{x})$$

and

$$L(\mathbf{x}+\mathbf{y}) = L\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix} = \begin{pmatrix} 2(x_2+y_2) \\ 4(x_1+y_1) \end{pmatrix} = \begin{pmatrix} 2x_2 \\ 4x_1 \end{pmatrix} + \begin{pmatrix} 2y_2 \\ 4y_1 \end{pmatrix} = L(\mathbf{x}) + L(\mathbf{y})$$

for all vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^3$ and $\alpha \in \mathbb{R}$.

6. Let L be the linear operator on $\mathbb{R}^2$ given by

$$L(\mathbf{x}) = \begin{pmatrix} x_1 - x_2 \\ x_2 \end{pmatrix}.$$

a) Write down the matrix A representing L with respect to the standard basis for $\mathbb{R}^2$.

We calculate the columns of A:

$$\mathbf{a}_1 = L\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\mathbf{a}_2 = L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Therefore

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

b) Write down the matrix B representing L with respect to the basis

$$F = [\mathbf{u}_1, \mathbf{u}_2] = \left[\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 3 \end{pmatrix}\right].$$

The transition matrix to change from the basis F to the standard basis is

$$S = \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}.$$

Therefore

$$B = S^{-1}AS = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -2 & -9 \\ 1 & 4 \end{pmatrix}.$$

Note that $S^{-1}A = \begin{pmatrix} -2 & -7 \\ 1 & 3 \end{pmatrix}$ would be the matrix representing L as a linear transformation from $\mathbb{R}^2$ with the basis F to $\mathbb{R}^2$ with the standard basis. The matrix B is the matrix representing L as a linear transformation from $\mathbb{R}^2$ with the basis F to $\mathbb{R}^2$ with the basis F. For instance, the second column of $S^{-1}A$ is

$$L(\mathbf{u}_2) = \begin{pmatrix} -7 \\ 3 \end{pmatrix} \text{ wrt the standard basis.}$$

To get the second column of B we have to rewrite this wrt the basis F. We find that

$$\begin{pmatrix} -7 \\ 3 \end{pmatrix} \text{ wrt the standard basis } = -9\mathbf{u}_1 + 4\mathbf{u}_2$$

and thus the second column of B is $\begin{pmatrix} -9 \\ 4 \end{pmatrix}$.

[EXTRA CREDIT]

7. a) Find the transition matrix corresponding to the change of basis from

$$[\mathbf{u}_1, \mathbf{u}_2] = \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix} \right] \qquad \text{to} \qquad [\mathbf{v}_1, \mathbf{v}_2] = \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix} \right].$$

The transition matrix from $[\mathbf{u}_1, \mathbf{u}_2]$ *to the standard basis is* $U = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ *and the transition matrix from* $[\mathbf{v}_1, \mathbf{v}_2]$ *to the standard basis is* $V = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$*. Therefore the required transition matrix is*

$$S = V^{-1}U = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} = \begin{pmatrix} -7 & -19 \\ 3 & 8 \end{pmatrix}.$$

b) If $\mathbf{x} = 3\mathbf{u}_1 - \mathbf{u}_2$, find the coordinates of $\mathbf{x}$ with respect to $[\mathbf{v}_1, \mathbf{v}_2]$.

The coordinates of $\mathbf{x}$ *wrt* $[\mathbf{u}_1, \mathbf{u}_2]$ *are* $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$*. Therefore the coordinates wrt to* $[\mathbf{v}_1, \mathbf{v}_2]$ *are*

$$S \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -7 & -19 \\ 3 & 8 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

i.e. $\mathbf{x} = -2\mathbf{v}_1 + \mathbf{v}_2$*. To check this, you could calculate both* $3\mathbf{u}_1 - \mathbf{u}_2$ *and* $-2\mathbf{v}_1 + \mathbf{v}_2$ *wrt the standard basis; in both cases you would get* $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$*. However, there is no reason to calculate* $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *at the beginning of the problem, and certainly* $S\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *won't give you anything meaningful. The whole point of* S *is that it changes coordinates wrt to* $[\mathbf{u}_1, \mathbf{u}_2]$ *to coordinates wrt to* $[\mathbf{v}_1, \mathbf{v}_2]$*, without any need to write the vector wrt the standard basis.*