Mar 27

Note Title 3/27/2009

Tool for working with alt. series —

Suppose $\sum_{i=1}^{\infty} c_i$ is alternating i.e. each successive term has the opposite sign of the previous one.

* suppose $|c_i|$ decrease.

Then for all $k \in \mathbb{N}$,

$$0 \leq \left| \sum_{i=1}^{k} c_i \right| \leq |c_1|$$

proof

Case 1 assume $c_1 > 0$.

If $\overline{k \text{ - odd}}$ -

$$\sum_{i=1}^{k} c_i = c_1 + (c_2 + c_3) + (c_4 + c_5) + \cdots + (c_{k-1} + c_{2k}).$$

$$= |C_1| + \underbrace{(-|C_2| + |C_3|)}_{\leq 0}$$

$$+ \underbrace{(-|C_4| + |C_5|)}_{0 \geq} \cdots + \underbrace{(-|C_{k-1}| + |C_k|)}_{0 \geq}$$

$$\leq |C_1|$$

$$\sum_{i=1}^{k} C_i = (C_1 + C_2) + (C_3 + C_4)$$

$$+ \cdots + (C_{k-2} + C_{k-1}) + C_k$$

$$= \underbrace{(|C_1| - |C_2|)}_{\geq 0} + \underbrace{(|C_3| - |C_4|)}_{0 \geq}$$

$$+ \cdots + \underbrace{(|C_{k-2}| - |C_{k-1}|)}_{\geq 0} + \underbrace{|C_k|}_{\geq 0}$$

$$\geq 0$$

$$0 \leq \left| \sum_{i=1}^{k} c_i \right| \leq |c_1|$$

k even –

$$\sum_{i=1}^{k} c_i = c_1 + (c_2 + c_3) + (c_4 + c_5) + \cdots + (c_{k-2} + c_{k-1}) + c_k$$

$$= |c_1| + (\underbrace{-|c_2| + |c_3|}_{\leq 0}) + (\underbrace{-|c_4| + |c_5|}_{\leq 0}) + \cdots + (\underbrace{-|c_{k-2}| + |c_{k-1}|}_{\overset{?}{\leq} 0})$$

$-|C_k| \leq 0$

$\leq |C_1|$

$$\sum_{i=1}^{k} c_i = (C_1 + C_2) + (C_3 + C_4) + \cdots + (C_{k-1} - C_k)$$

$$= \underbrace{(|C_1| - |C_2|)}_{0 \leq} + (|C_3| - |C_4|) + \cdots + \underbrace{(|C_{k-1}| - |C_k|)}_{0 \leq}$$

≥ 0

$$0 \leq \left| \sum_{i=1}^{k} c_i \right| \leq |c_1|$$

<u>Case 2</u> $\quad c_1 \leq 0$

Consider

$$d_i = -c_i$$

Now $\sum_{i=1}^{\infty} d_i$ satisfies

<u>case 1</u> — So

$$0 \leq \left| \sum_{i=1}^{k} d_i \right| \leq |d_1|$$

$$0 \leq \left| \sum_{i=1}^{k} (-c_i) \right| \leq |c_1|$$

$$0 \le \left| - \sum_{i=1}^{k} c_i \right| \le |c_1|$$

$$0 \le \left| \sum_{i=1}^{k} c_i \right| \le |c_1|$$

Suppose I have an alt series $\sum_{i=1}^{\infty} c_i$ & $|c_i| \searrow$.

Then for all

$n \le m$

$$\left|\sum_{i=n}^{m} c_i\right| \le |C_n|$$

Ex. Show

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{x^2+n}$$

Converges uniformly on all of $\mathbb{R}$.

Show it is uniformly Cauchy i.e. for $\varepsilon > 0$ I will find a N so that if $m \geq n \geq N$ then $\forall x$, $\left| \sum_{i=n}^{m} \left(\frac{(-1)^{i-1}}{x^2+i} \right) \right| < \varepsilon$.

$$\left| \sum_{i=n}^{m} \frac{(-1)^{i-1}}{x^2+i} \right| \leq \frac{1}{x^2+n} \leq \frac{1}{n}$$

So for $\varepsilon > 0$, select

$N > \frac{1}{\varepsilon}$ & now

for all $m \geq n \geq N$

$$\left| \sum_{i=n}^{m} \frac{(-1)^{i-1}}{x^2+i} \right| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

Ex $\sum_{n=1}^{\infty} \frac{x^n}{n^2+n+1}$, conv on $[-1,1]$

diff. Termwise $\sum_{n=1}^{\infty} \left(\frac{x^{n-1}}{n+1+\frac{1}{n}} \right)$

$|x| < 1$ - <u>conv</u>

on $[-1+\delta, 1-\delta]$ it

conv. <u>unif</u>.

bded by $|x^{n-1}|$

What happens at <u>+1</u>?

<u>diverges</u>

What about -1?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+1+\frac{1}{n}} \quad - \quad \underline{\text{conv.}}$$

Is this conv. unif.

on $[-1, -1+\delta]$

$$\sum_{n=1}^{\infty} \left(\frac{x^{n-1}}{n+1+\frac{1}{n}} \right)$$

$$x \in [-1, -1+\delta]$$

$$0 < \delta < \frac{1}{2}.$$

so $x < 0$ &

series always

alternates

is. $\frac{1}{(n+1)+1+\frac{1}{n+1}} < \frac{1}{n+1+\frac{1}{n}}$?

$$\iff (\cancel{n}+1)+\cancel{1}+\frac{1}{n+1} > \cancel{n}+\cancel{1}+\frac{1}{n}$$

$$\underline{1+\frac{1}{n+1}} > \frac{1}{\underset{\uparrow}{n}} \quad \underline{\underline{\text{yes}}}$$

<u>So</u> Can apply my alt. series notion —

$$\sum_{i=1}^{\infty} \frac{x^{i-1}}{i+1+\frac{1}{i}} \quad \text{on } [-1, -1+\delta]$$

Show uniformly Cauchy

$n \leq m$

$$0 \leq \left| \sum_{i=n}^{m} \frac{x^{i-1}}{i+1+\frac{1}{i}} \right| \leq \frac{|x|^{n-1}}{n+1+\frac{1}{n}}$$

$$\leq \frac{1}{n}$$

For $\varepsilon > 0$, select N

so that $N > \frac{1}{\varepsilon} \Rightarrow \frac{1}{N} < \varepsilon$

Now if $m \geq n \geq N$

then $\left| \sum_{i=n}^{m} \frac{x^{i-1}}{i+1+\frac{1}{i}} \right| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$

+ So the sum of deriv. conv. unif on $[-1, -1+\delta]$.

$$\sum_{n=1}^{\infty} \frac{n x^n}{2^{n+3}}$$

$$= \sum_{n=1}^{\infty} \frac{n}{8} \left(\frac{x}{2}\right)^n$$

I know if $|x| \geq 2$ - diverges

as terms don't $\to 0$

$$|x| < 2.$$

$$\sum_{n=1}^{\infty} \frac{n}{8} \left(\frac{x}{2}\right)^n$$

$$\left|\frac{x}{2}\right| < 1$$

let $r = \left|\frac{x}{2}\right|$

Look at $\displaystyle\sum_{n=1}^{\infty} \frac{n}{8} r^n$

$$= \sum_{n=1}^{\infty} \underline{\underline{\frac{n}{8}(\sqrt{r})^n}} \underline{(\sqrt{r})^n}$$

$$(\sqrt{r} < 1)$$

Now $\lim_{n \to \infty} \frac{n}{8}(\sqrt{r})^n = 0$

$$\frac{n}{(\sqrt{r})^{-n}} \quad \underline{\text{L'hôpital}}$$

$$\lim_{n \to \infty} \frac{1}{-n\sqrt{r}^{-n} \log(\sqrt{r})} = 0$$

$\Rightarrow$ as $\frac{n}{8}(\sqrt{r})^n$ con<u>verg</u>es it is <u>bounded</u>.

so $\exists B$ & $\forall n$

$$\frac{n}{8}(\sqrt{r})^n \le B$$

& $$\left|\frac{nx^n}{2^{n+3}}\right| \le B\left|\frac{x}{\sqrt{2}}\right|^n$$

& This gives the control we <u>need</u>.