Mar 23

Note Title 3/23/2009

1) a) $f_k(x) = x^k$ on $[0,1]$.

$$\lim_{k\to\infty} x^k = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$$

why? i) $x=1$, clear.

ii) $0 \leq x < 1$.

x^k - decreases

$0 \leq x^{k+1} = x \cdot x^k < x^k$

$\Rightarrow$ Converges —

$$\overline{|x^k - 0|} \overset{?}{<} \varepsilon$$

$$x^k \overset{?}{<} \varepsilon$$

$$\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$$

so as series

conv. $\lim_{k\to\infty} x^k = 0$

Let $\lim_{k\to\infty} x^k = a$

$$a = \lim_{k\to\infty} x \cdot x^k = x \cdot a$$

$$\lim_{k\to\infty} x^{k+1}$$

$$a = xa$$

$$(x-1)a = 0, \quad x-1 \neq 0$$

$$\Rightarrow a = 0$$

Can conv. be unif.?

NO

Because limit is discontinuous & a unif. limit of cont fctns is cont.

Delete bad pt.

1) $[0,1)$

Let's look at

$$\sup_{x\in[0,1)}(x^k) = 1$$

$$\not\to 1 \not\to 0$$

$$\sup_{x\in[0,1)}\left(|x^k - 0|\right) \not\to 0 \text{ in } k$$

But remove a small interval.

$[0, 1-\delta]$, $\delta > 0$.

Here at all x, $x^k \to 0$.

Look at

$$\|x^k - 0\|_{\sup} = \sup_{x \in [0,1-\delta]} (|x^k - 0|) = \sup_{x \in [0,1-\delta]} x^k$$

$$= (1-\delta)^k$$

$$\lim_{k \to \infty} (1-\delta)^k = 0$$

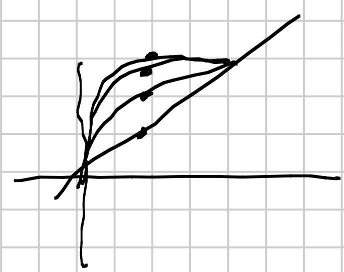
so on $[0, 1-\delta]$, $x^k \to 0$ uniformly

b) $x^{1/k}$, $[0,1]$

$$\lim_{k \to 0} x^{1/k} = \begin{cases} 0, & x = 0 \\ 1, & x \in (0,1] \end{cases}$$

i) $x = 0$, <u>clear</u>

ii) $[0,1]$. ?



$$x^{1/k}$$

$$\log(x^{1/k})$$

$$= \frac{1}{k} \log(x)$$

as $k \to \infty$, $\frac{1}{k} \log(x) \to 0$

$$= \log(\lim x^{1/k})$$

$$\Rightarrow \lim_{k \to \infty} x^{1/k} = 1$$

SO - have "computed" the limit. -

Is it Unif?

How about on $[0,1]$?
Same problem -

$[\delta, 1]$, $\delta > 0$?

is it uniform?

$$\sup_{x \in [\delta, 1]} \left(\left| x^{1/k} - 1 \right| \right)$$

$x^{1/k}$ is increasing & always ≤ 1.

$$\left| x^{1/k} - 1 \right| = 1 - x^{1/k}$$

is a noneg- & dec. fctn.

so sup is at δ.

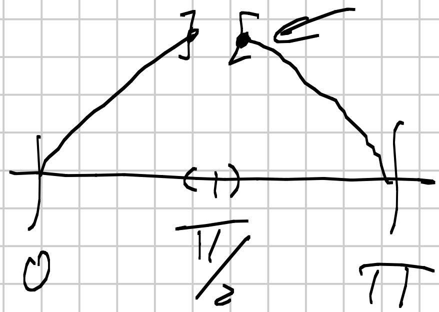
+ is $(1-\delta^{1/k})$

+ $\lim_{k\to\infty}(1-\delta^{1/k}) = 0$.

c) $\sin^k(x) = \cancel{\sin(\sin(\sin}\cdots(\sin(x))) \cdot)$

$\sin(x)^k$ $\quad [0,\pi]$

$\pi/2, \longrightarrow 1$



a) $f_k(x) = \dfrac{e^{-x^2/k}}{k}$ $\quad x \in \mathbb{R}$.

1) $f_k(x) > 0$.

2) $f_k' = -\dfrac{2x}{k^2} e^{-x^2/k}$

< 0 if $x > 0$

> 0 if $x < 0$

$= 0$ at $x = 0$

$\Rightarrow$ For $x > 0$, dec.

For $x < 0$, inc.

f has a unique max at $\underline{0}$.

$$0 \leq f_k(x) \leq f_k(0) = \frac{1}{k}$$

<u>so</u> $\|f_k(x)\|_{\sup} = \frac{1}{k}$.

so $f_k \to 0$ uniformly

2) $kxe^{-kx} = f_k(x)$, $x \in [0, \infty)$

$f_k(x) \geq 0$

$$f_k'(x) = -k^2xe^{-kx} + ke^{-kx}$$

$$= 0$$

line - decreases

$$-k^2x + k = 0$$

$$x = \frac{1}{k}$$

$x < \frac{1}{k}$, $f_k' > 0$

$x > \frac{1}{k}$, $f_k' < 0$

so f_k increases on $[0, \frac{1}{k})$ &

decreases on $(\frac{1}{k}, \infty)$

& so has a unique max at $\frac{1}{k}$.

$$\boxed{\& \quad f_k\left(\frac{1}{k}\right) = \frac{1}{e}.}$$

But what is

$$\lim_{k \to \infty} kx e^{-kx} ?$$

$$\lim_{k \to \infty} \frac{kx}{e^{kx}} \quad \text{— L'Hôpital —}$$

$\underline{\underline{x \neq 0}}$

$$\lim_{k \to \infty} \frac{\cancel{x}1}{\cancel{x}e^{kx}} = 0$$

So limit is 0.

$$\sup_{x \in [0,\infty)} |f_k(x)| = \frac{1}{e}$$

Not Unif.

HW
p 319 #2
p 322 #2
#4