Mar 13

Note Title 3/13/2009

Uniform Conv.

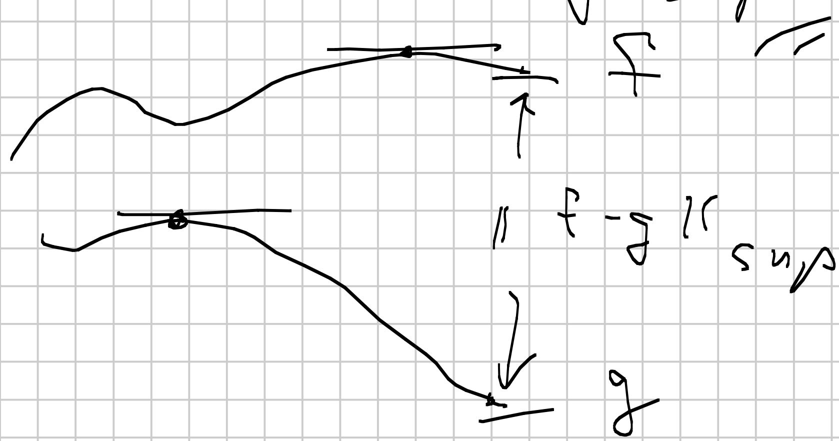
& Integration —

$$\|f\|_{sup} = \sup_{x \in S} |f(x)|$$

$$f, g : S \to \mathbb{R},$$

$$\left| \sup_{x \in S} f(x) - \sup_{x \in S} g(x) \right|$$

$$\leq \|f-g\|_{sup} \quad //$$



$$\left| \inf_{x \in S} f - \inf_{x \in S} g \right|$$

$$\leq \|f-g\|_{sup}$$

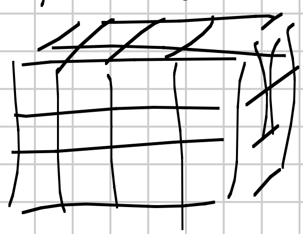
Integration in $\mathbb{R}^k$.

f - bounded on $S = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_k, b_k]$

"Partition" S.

Take partitions P_i of $[a_i, b_i]$

$$\tilde{P} = P_1 \times P_2 \cdots \times P_n$$



little <u>cells</u>

$$C_1, C_2, \ldots, C_L$$

$$M_i = \sup_{x \in C_i} f(x)$$

$$m_i = \inf_{x \in C_i} f(x)$$

$$U(f,\vec{P}) = \sum_{i=1}^{r} M_i \operatorname{vol}(C_i)$$

$$L(f,\vec{P}) = \sum_{i=1}^{r} m_i \cdot \operatorname{vol}(C_i).$$

$$U(f) = \inf_{\vec{P}} (U(f,P))$$

$$L(f) = \sup_{\vec{P}} (L(f,P))$$

$$U(f) \geq L(f)$$

f is R. int.

iff $U(f) = L(f)$

$$+ \int_S f \, d\vec{x} = U(f) = L(f)$$

If f, g are bounded on S & $\overline{P}$ is any partition then

$$|U(f, P) - U(g, P)| \leq \|f - g\|_{\sup} vol(S).$$

$$|L(f,P) - L(g,P)| \leq \|f-g\|_{sup} \, vol(S).$$

Pf.

$$\left| \sum_{i=1}^{L} M_i(f) \, vol(C_i) - \sum_{i=1}^{L} M_i(g) \, vol(C_i) \right|$$

$$\leq \sum_{i=1}^{L} |\underline{M}_i(f) - \underline{M}_i(g)| \, vol(C_i)$$

$$\leq \sum_{i=1}^{L} \|f-g\|_{sup} \, vol(C_i)$$

$$= \|f-g\|_{\sup} \sum_{i=1}^{L} \mathrm{vol}(C_i)$$

$$= \|f-g\|_{\sup} \mathrm{vol}(S).$$

<u>Lemma</u> Suppose

$f_k \to f$ <u>unif.</u>

& f_k are R-int.

<u>Then f</u> is R-int.

Pf Let $\varepsilon > 0$.

As S_k are R. int.

$\exists$ part. $\vec{P}_k$,

with $|U(f_k, \vec{P}_k) - L(f_k, \vec{P}_k)|$

$$< \frac{\varepsilon}{4}$$

& $\exists k$ so that

$$\| f - f_k \|_{sup} < \frac{\varepsilon}{4 vol(S)}$$

For this k look at

$$|U(f, P_k) - L(f, P_k)|$$

$$\leq |U(f, P_k) - U(f_k, P_k)|$$

$$+ |U(f_k, P_k) - L(f_k, P_k)|$$

$$+ |L(f_k, P_k) - L(f, P_k)|$$

$$\leq \|f - f_k\|_{sup} \, vol(S) + \varepsilon/4$$

$$+ \|f - f_k\|_{sup} \, vol(S).$$

$< \frac{3\varepsilon}{4} < \varepsilon$.

$\Longrightarrow U(f) = L(f)$
& f is int'eg.

Thm If f_k are R int,
$f_k \to f$ unif. then
f is R int. &

$$\int_S f \, d\vec{x} = \lim_{k\to\infty} \int f_k \, d\vec{x}$$

Pf

$$\left| \int_S f \, d\vec{x} - \int_S f_k \, d\vec{x} \right|$$

$$= \left| \int_S f - f_k \, d\vec{x} \right|$$

$$\leq \int_S |f - f_k| \, d\vec{x}$$

$$\leq \int_S \sup_{\vec{x} \in S} |f(\vec{x}) - f_k(\vec{x})| \, dx$$

$$= \|f - f_k\|_{\sup} \operatorname{vol}(S)$$

So

$$\lim_{k \to \infty} \left| \int_S f \, dx - \int_S f_k \, dx \right|$$

$$\leq \lim_{k \to \infty} \|f - f_k\|_{\sup} \operatorname{vol}(S)$$

$$= 0$$

Thm Suppose f_k of class C^1 on $[a,b]$.

& 1) f_k' conv. unif. to g.

2) $f_k(a) \xrightarrow[k]{} L$.

Then f_k conv. unif. to some f, f is C^1 & $f' = g$.

Pf. f_k' are cont.

$\Rightarrow g$ is cont.

Now $f_k(x) = f_k(a) + \int_a^x f_k'(t)\,dt$

F.T.

& $\|f_n - f_m\|_{sup}$

$$= \sup_{x\in[a,b]} \Big| f_n(a) - f_m(a) + \int_a^x (f_n'(t) - f_m'(t))\,dt \Big|$$

$$\leq \sup_{x \in [a,b]} \left(|f_n(a) - f_m(a)| + \int_a^x |f_n'(t) - f_m'(t)|\, dt \right)$$

$$\leq |f_n(a) - f_m(a)| + \int_a^x \|f_n' - f_m'\|_{\sup}\, dt$$

$$\leq |f_n(a) - f_m(a)| + \|f_n' - f_m'\|_{\sup} (b-a)$$

$\forall \varepsilon > 0, \ \exists K, \ n, m \geq K$

$$|f_n(a) - f_m(a)| < \varepsilon/2 \ \forall$$

$$\|f_n' - f_m'\|_{sup} < \frac{\varepsilon}{2(b-a)}$$

$$\Rightarrow \|f_n - f_m\|_{sup} < \underline{\underline{\varepsilon}}$$

$\Rightarrow f_n$ are <u>unif.</u> Cauchy &

so conv. <u>unif.</u>

<u>So</u> $f_n \to f$ <u>uni</u>f

Notice

$$f_n(x) = f_n(a) + \int_a^x f_n'(t)\,dt$$

Let $n \to \infty$ ↓ By Int, Thm

$$f(x) = L + \int_a^x g(t)\,dt$$

$\Rightarrow$ f is diff. &

$$f'(x) = g(x) \quad (F.T.)$$

$f_k(x) = k$ on $[0,1]$

$f_k' = 0 \rightarrow$ unif