Jan 26

Note Title 1/26/2009

Parametrized Curves

arc length

$\mathbb{R}^k$, $k > 1$

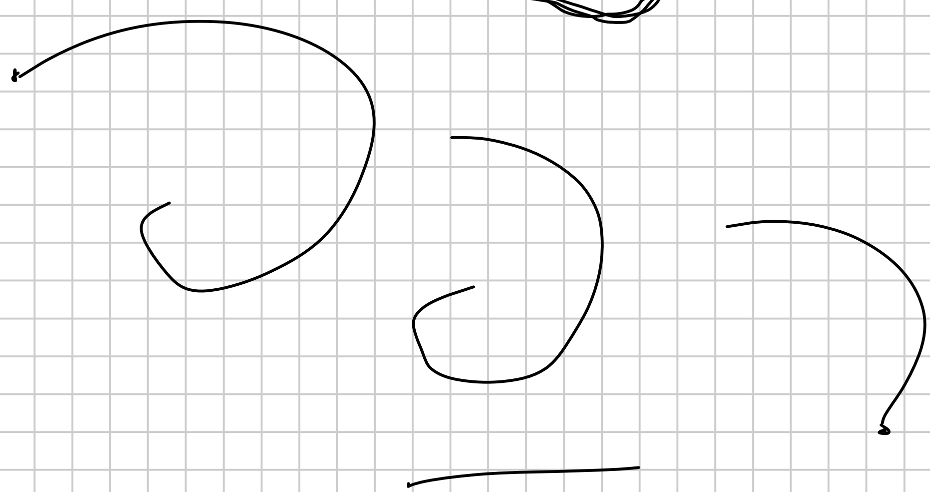
$g : [a, b] \to \mathbb{R}^k$

1) C^1, $g'(t) \neq 0$.

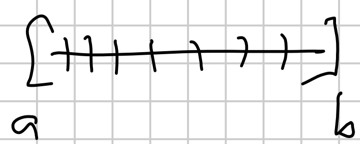
$g(t) = (g_1(t), g_2(t), \ldots, g_k(t))$

$g'(t) = (g_1'(t), g_2'(t), \ldots, g_k'(t))$

2) g - 1-1.



? what is the "length" of $g([a,b])$?



$$P = \{a = x_0 < x_1 \cdots x_n = b\}.$$

$$g(x_0), g(x_1) \cdots g(x_n).$$

Approx.

$$= \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|$$

$$\sqrt{\underbrace{(g_1(x_i)-g_1(x_{i-1}))^2}+\cdots+(g_k(x_i)-g_k(x_{i-1}))^2}$$

$$g_1(x_i)-g_1(x_{i-1}) = g_1'(y_{1,i})(x_i - x_{i-1})$$

MVT $$g_2(x_i)-g_2(x_{i-1}) = g_2'(y_{2,i})(x_i - x_{i-1})$$

$$\vdots$$

$$g_k(x_i)-g_k(x_{i-1}) = g_k'(y_{k,i})(x_i - x_{i-1})$$

$$= \sqrt{g_1'(y_{1,i})^2 + g_2'(y_{2,i})^2 + \cdots + g_k'(y_{k,i})^2}\,(x_i - x_{i-1})$$

$g_1' \cdots g_k'$ are cont.
on $[a,b]$.
All unif. cont.
$\Rightarrow$ for $\varepsilon > 0$, $\exists \delta$ + if
$|x_i - x_{i-1}| < \delta$ then
$$|g_j'(y) - g_j'(x_i)| < \varepsilon$$
$\forall y \in [x_{i-1}, x_i]$.

Assume $\operatorname{Mesh}(P) < \delta$.

+ Now

$$\simeq \left(\sqrt{g_1'(x_i)^2 + g_2'(x_i)^2 \cdots + g_k'(x_i)^2}\,(x_i - x_{i-1})\right.$$

$$\left.\pm k\varepsilon(x_i - x_{i-1})\right)$$

$$= |g'(x_i)|(x_i - x_{i-1}) \pm k\varepsilon(x_i - x_{i-1}).$$

$$\overbrace{\sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|}$$

$$= \sum_{i=1}^{n} |g'(x_i)|(x_i - x_{i-1})$$

$$\pm \sum_{i=1}^{n} k\varepsilon (x_i - x_{i-1})$$

$$= \underbrace{\sum_{i=1}^{n} |g'(x_i)|(x_i - x_{i-1})}_{\text{R. S. for } \int_a^b |g'(t)|\,dt.} \boxed{\pm k\varepsilon (b-a)}$$

As $\text{Mesh}(P) \to 0$

$$\sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|$$

$$\longrightarrow \int_a^b |g'(t)|\,dt.$$

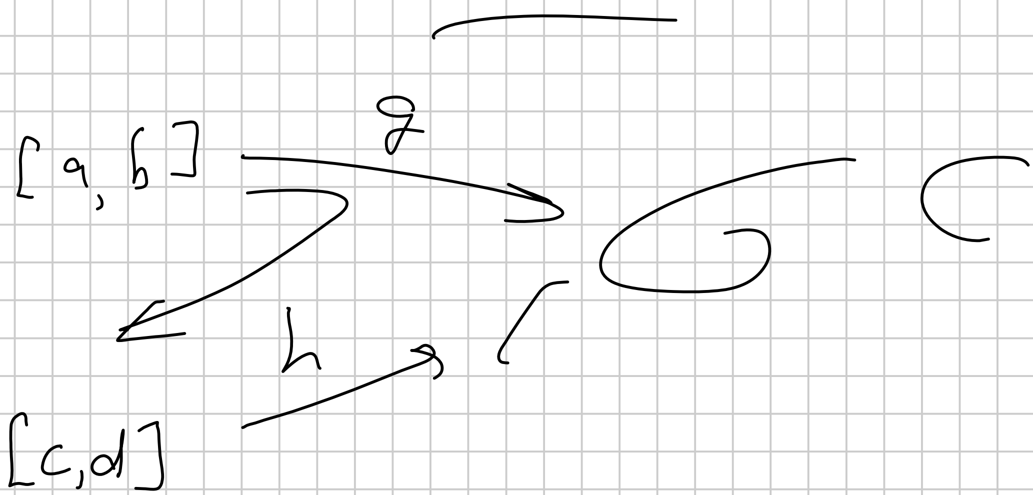
arc length

$|g'(t)|$ - speed.

Thm Suppose $g : [a,b] \to \mathbb{R}^k$ & $h : [c,d] \to \mathbb{R}^k$ both give 1-1 & C^1-param. of the same curve.

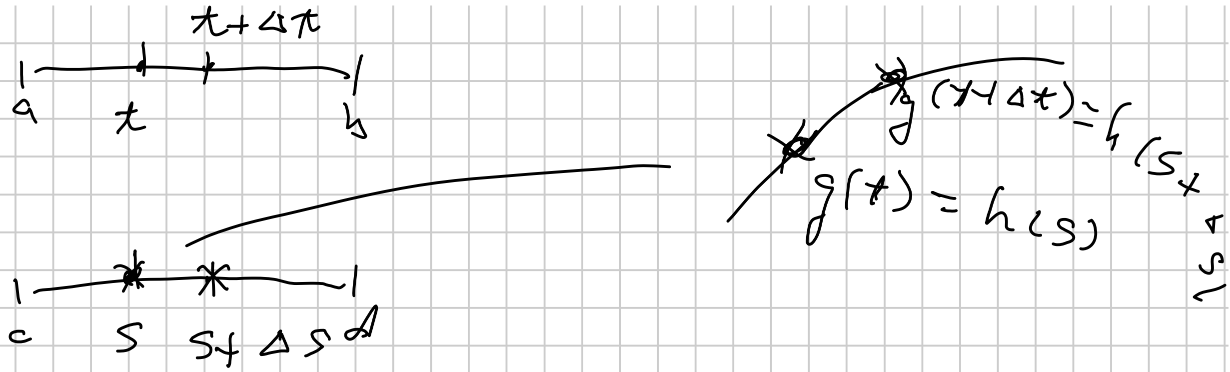
3

Then $\int_a^b |g'(t)|\,dt = \int_c^d |h'(s)|\,ds.$



$$\varphi = h^{-1} g : [a,b] \to [c,d].$$

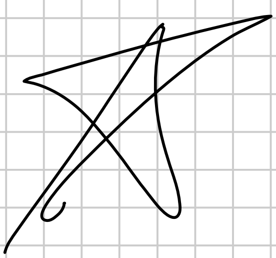
What is φ'?



$$g(t+\Delta t) = g(t) + \underbrace{g'(t)\,\Delta t + e(\Delta t)}$$

$$\qquad\; \| \qquad\qquad\quad \|$$

$$h(s+\Delta s) = h(s) + \underbrace{h'(s)\,\Delta s + \hat{e}(\Delta s)}$$



$$, \quad g'(t)\,\Delta t + e(\Delta t) = h'(s)\,\Delta s + \hat{e}(\Delta s)$$

$$g'(t) + \underbrace{\cancel{\frac{e(\Delta t)}{\Delta t}}}_{0} = h'(s)\,\frac{\Delta s}{\Delta t} + \frac{\hat{e}(\Delta s)}{\Delta t}$$

$$= \left(h'(s) + \underbrace{\frac{\hat{e}(\Delta s)}{\Delta s}}_{0}\right)\frac{\Delta s}{\Delta t}$$

Let $\Delta t \to 0$

g + h are cont.

So as $\Delta t \to 0$,

$\Delta s \to 0$

$$\frac{e(\Delta t)}{\Delta t} \to 0$$

$g'(t)$ $\qquad\qquad$ $h'(s)$

as $\Delta t \to 0$, $\frac{\Delta s}{\Delta t}$ must conv.

$g'(t)$ + $h'(s)$ are //.

+ the ratio of

their lengths is $\lim_{t \to 0} \frac{\Delta s}{\Delta t}$

so

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{|g'(t)|}{|h'(\underbrace{h^{-1} g(t)}_{s})|}$$

$$\int_a^b |g'(t)|\,dt \qquad g(t) = h(\varphi(t))$$

$$= \int_a^b |h'(\varphi(t))\,\varphi'(t)|\,dt$$

$$= \int_a^b |h'(\varphi(t))|\,|\varphi'(t)|\,dt$$

Case 1 $\underline{\underline{\varphi' > 0}}$

$$= \int_a^b |h'(\varphi(t))|\, \varphi'(t)\, dt$$

$$s = \varphi(t)$$

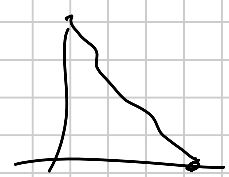
$$ds = \varphi'(t)\, dt$$

$$\int_{\varphi(a)}^{\varphi(b)} |h'(s)|\, ds = \int_c^d |h'(s)|\, ds.$$

$$\varphi' < 0 \qquad = \int_a^b |g'(t)|\, dt$$

$$\int_a^b |h'(\varphi(t))|\, |\varphi'(t)|\, dt$$

$$= -\int_a^b |h'(\varphi(t))\, \varphi'(t)\, dt$$



Sec 5.1 #1, 6

$$= -\int_{\varphi(a)}^{\varphi(b)} |h'(s)|\, ds$$

$$= -\int_d^c |h'(s)|\, ds = \int_c^d |h'(s)|\, ds.$$