Jan 23

Note Title 1/23/2009

Brief overview of Measure Theory

1) Measurable set -

2) Measure

$\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, Length Area Volume

X

Measurable sets

1) open + closed sets are msble.

2) If A is msble so is A^c.

3) If $A_1, A_2, A_3, \ldots,$ all msble then $\bigcup_{i=1}^{\infty} A_i$ + $\bigcap_{i=1}^{\infty} A_i$ are msble.

$$\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i^c \right)^c$$

$\boxed{\sigma\text{-algebra}}$

$\mathbb{Q}$ is msble –

$\{q\}$ is msble.

Rationals are countable

$\mathbb{Q} = \{q_1, q_2 \ldots\}$.

$$\mathbb{Q} = \bigcup_{i=1}^{\infty} \{q_i\}$$

$\mathcal{B}$.

Measure:

$$m : \mathcal{B} \to \mathbb{R}^+ \cup \{\infty\}.$$

1) $m(\emptyset) = 0$

2) If $A = \begin{cases} \text{int} \\ \text{rect.} \\ \text{box} \end{cases}$, $m(A) = \begin{cases} \text{length} \\ \text{area} \\ \text{vol.} \end{cases}$

3) If $A_1, A_2 \cdots \in \mathcal{B}$

$\quad \forall\ A_i \cap A_j = \emptyset$, $i \neq j$

then $m\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} m(A_i)$.

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4) If $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$, in $\mathcal{B}$.

then $m\left(\bigcup_{i=1}^{\infty} B_i\right) = \lim_{i\to\infty} \mu(B_i)$.

Continuity from below.

5) If $B_1 \supseteq B_2 \supseteq \cdots$

& $m(B_1) < \infty$, then

$$m\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_{i\to\infty} m(B_i)$$

Cont. from above

Let $B_i = [i, \infty)$

$m(B_i) = \infty$

$\bigcap_{i=1}^{\infty} B_i = \emptyset$, $m(\emptyset) = 0$

Show 1), 2), 3) $\Rightarrow$ 4)

i) Suppose $A \subseteq B$, in $\mathcal{B}$.

Then $m(A) \leq m(B)$.

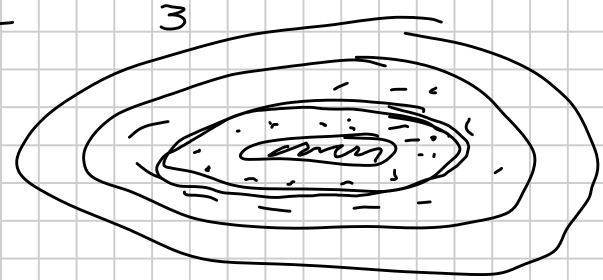
$B = A \cup \underset{\overset{\parallel}{B \cap A^c}}{(B - A)}$

$$m(B) = m(A) + m(B-A)$$

by 3)

$$\geq m(A)$$

4) $B_1 \subseteq B_2 \subseteq B_3 \ldots$



$A_1 = B_1$

$A_2 = B_2 - B_1$

$A_3 = B_3 - B_2$

$\vdots$

$A_n = B_n - B_{n-1}$

$$B_n = \bigcup_{i=1}^{n} A_i$$

A_i - disjoint

$$m(B_n) = \sum_{i=1}^{n} m(A_i).$$

$$\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i.$$

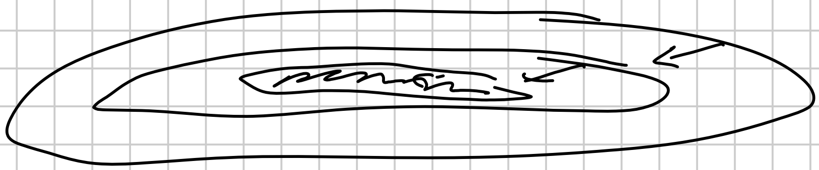
$$m\left(\bigcup_{i=1}^{\infty} B_i\right) = m\left(\bigcup_{i=1}^{\infty} A_i\right)$$

$$= \sum_{i=1}^{\infty} m(A_i) = \lim_{n\to\infty} \sum_{i=1}^{n} m(A_i)$$

$$= \lim_{n\to\infty} m(B_n).$$

4) $\Rightarrow$ 5)

$$B_1 \supseteq B_2 \supseteq B_3 \dots$$



$$C_i = B_1 - B_i = B_1 \cap B_i^c$$

$$C_1 \subseteq C_2 \subseteq C_3 \ldots$$

$$\bigcup_{i=1}^{\infty} C_i = \bigcup_{i=1}^{\infty} (B_1 \cap B_i^c)$$

$$= B_1 \cap \bigcup_{i=1}^{\infty} (B_i^c)$$

$$= B_1 \cap \left(\bigcap_{i=1}^{\infty} B_i\right)^c$$

$$= B_1 - \left(\bigcap_{i=1}^{\infty} B_i\right)$$

$$m\left(B_1 - \bigcap_{i=1}^{\infty} B_i\right) = m\left(\bigcup_{i=1}^{\infty} C_i\right)$$

$$= \lim_{i\to\infty} m(C_i) \quad \text{by } 4)$$

$$= \lim_{i\to\infty} m(B_1 - B_i)$$

? is $m(B_1 - B_i) = m(B_1) - m(B_i)$?

it is if $m(B_1) < \infty$.

i) If $A \subseteq B$ & $m(B) < \infty$

then $m(B - A) = m(B) - m(A)$.

$$\left(m(B-A) + m(A) = m(B) \right)$$

all #'s finite.

$$\cancel{m(B_1)} - m\left(\bigcap_{i=1}^{\infty} B_i\right)$$

$$= \cancel{m(B_1)} - \lim_{n\to\infty} m(B_n)$$

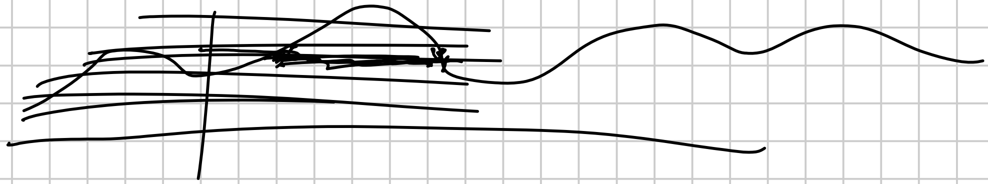
Integration:

Measurable functions.

Means $\forall\, a \in \overline{\mathbb{R}}$

$$\{x \mid f(x) > a\} \in \mathcal{B}.$$

Int. nonneg. fcts.



$$S_n(f) = \sum_{j=1}^{\infty} \frac{j}{2^n}\, m\left(\left\{x \mid \frac{j}{2^n} \le f(x) < \frac{j+1}{2^n}\right\}\right)$$

$$S_{n+1}(f) \le S_n(f)$$

$$\int f\, dm = \lim_{n\to\infty} S_n(f).$$

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}.$$

$$S_n(f) = 0.$$