Feb 18

Note Title 2/18/2009

Brief intro.
to exterior calculus
(Grassman Calculus)

Abstractly —

Exterior algebra –
"graded" algebra –
List of linear spaces

$A_0, A_1, A_3 \ldots A_n$

$(n+1)$ terms –

bases – coefficients

smooth functions

A_0 – smooth functions

(C^∞)

A_0 – 1 dim.

A_1 – n – dim.

elt. of A_1 looks like

$$\sum_{i=1}^{n} f_i \, \overline{|dx_i|}$$

$\nearrow$ coeff – smooth fctn.

A_2 – dim. $\binom{n}{2}$

elt look like

$$\sum_{1 \le i < j \le n} f_{i,j} \, \overline{|dx_i \wedge dx_j|}$$

A_k - dim $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$$\Rightarrow \underbrace{\sum_{\underbrace{1 \le i_1 < i_2 < \cdots < i_k \le n}} f_{(i_1 \ldots i_k)} \overbrace{dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}}}$$

$A_0 \quad A_1 \quad A_2 \quad \cdots \quad A_n$

$1 \quad n \quad \binom{n}{2} \quad \cdots \quad \binom{n}{n} = 1$

$n = 2, \quad 1 \quad 2 \quad 1$

$3 \quad 1 \quad 3 \quad 3 \quad 1$

$4 \quad 1 \quad 4 \quad 6 \quad 4 \quad 1$

Define "wedge product"

A_0 - 0-forms

A_1 - 1-forms

⋮

A_n - n-forms

$\alpha \in A_i$, $\beta \in A_j$

$\alpha \wedge \beta = A_{i+j}$

skew symmetric

$\alpha \wedge \beta = -\beta \wedge \alpha$

Ex.

$$\alpha = \sum_{i=1}^{n} f_i \, dx_i$$

$$\beta = \sum_{j=1}^{n} g_j \, dx_j$$

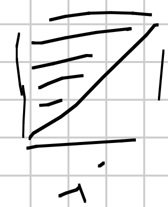
$$\alpha \wedge \beta = \left(\sum_{i=1}^{n} f_i \, dx_i\right) \wedge \left(\sum_{j=1}^{n} g_j \, dx_j\right)$$

(distributive
associative)

$$= \sum_{i,j=1}^{n} f_i \, g_j \, \underbrace{dx_i \wedge dx_j}$$

$$= \sum_{1 \le i < j \le n} \left(f_i \, g_j - f_j \, g_i\right) dx_i \wedge dx_j$$

In higher order products it gets more complicated



$$(dx_1 \wedge dx_3) \wedge (dx_2 \wedge dx_5)$$

$$dx_3 \wedge dx_2 = -dx_2 \wedge dx_3$$

$$= -dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5 .$$

Derivatives

$$A_0 \xrightarrow{d} A_1 \xrightarrow{d} A_2 \; \cdots \xrightarrow{d} A_n$$

$$\mathbb{R}^3 \quad \left(\text{fcts} \xrightarrow{\text{grad}} \text{v.F} \xrightarrow{\text{curl}} \text{v.F.} \xrightarrow{\text{div}} \text{fcts} \right)$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \cong \\ A_0 & A_1 & A_2 & \end{array}$$

$(dx_2 \wedge dx_3, \; \underline{dx_3 \wedge dx_1}, \; dx_1 \wedge dx_2)$

What is d?

<u>linear</u>

$$\omega = f \, dx_{i_1} \, dx_{i_2} \cdots dx_{i_k}$$

<u>k - form</u>

$$\begin{aligned} d\omega = \; & \frac{\partial f}{\partial x_1} dx_1 \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} + \\ & \frac{\partial f}{\partial x_2} dx_2 \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} + \\ & \vdots \\ & \frac{\partial f}{\partial x_n} dx_n \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \end{aligned}$$

Simplify

$f - A_0$

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \dots + \frac{\partial f}{\partial x_m} dx_m$$

$$= \mathrm{grad}(f).$$

$\mathbb{R}^3$ - 1-form

$$d(f_1\, dx_1 + f_2\, dx_2 + f_3\, dx_3)$$

$$\cancel{\frac{\partial f_1}{\partial x_1} dx_1 \wedge dx_1} + \frac{\partial f_1}{\partial x_2} \underline{dx_2 \wedge dx_1} + \frac{\partial f_1}{\partial x_3} dx_3 \wedge dx_1$$

$$+$$

$$\frac{\partial f_2}{\partial x_1} \underline{dx_1 \wedge dx_2} + \frac{\partial f_2}{\partial x_2} \cancel{dx_2 \wedge dx_2} + \frac{\partial f_2}{\partial x_3} dx_3 \wedge dx_2$$

$$+$$

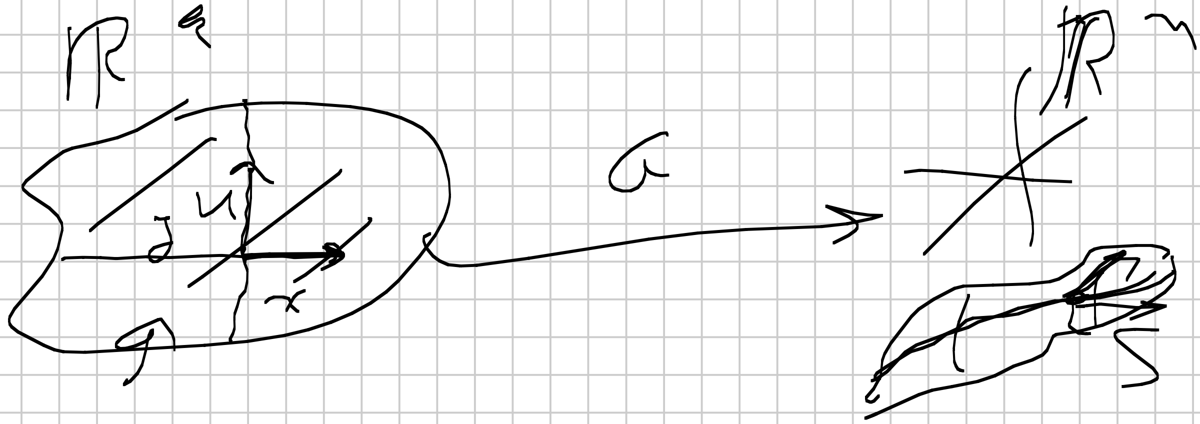
$$\frac{\partial f_3}{\partial x_1} dx_1 \wedge dx_3 + \frac{\partial f_3}{\partial x_2} \underline{dx_2} \wedge dx_3 + \frac{\partial f_3}{\partial x_3} \cancel{dx_3} \wedge dx_3$$

$$\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) \overbrace{dx_1 \wedge dx_2}^{(z)} + \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}\right) \overset{(x)}{dx_2 \wedge dx_3}$$

$$+ \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) \underset{(y)}{dx_3 \wedge dx_1}$$

curl

Forms are to be integrated over oriented smooth hypersurfaces.



G gives an orientation to S.

tangent vectors to images of lines // to standard basis vectors give a basis for tan. hyperplane.

ω — $\underline{k\text{-form}}$

$$\omega = f\, dx_{i_1} \wedge dx_{i_2} \cdots \wedge dx_{i_k}$$

$$\int_S f\, \underbrace{dx_{i_1} \wedge dx_{i_2} \cdots \wedge dx_{i_k}}$$

$k \times k$ matrix

$$= \int_U f(\underbrace{\underline{G}(u_1 \ldots u_k)}_{\text{basis for } \mathbb{R}^k}) \det\left(\frac{\partial G_{i_t}}{\partial u_j}\right) du_1\, du_2 \cdots du_k$$

(t, j) term of matrix

$k \times k$

Stoke's Thm

Suppose S is an oriented k-dim sfc in $\mathbb{R}^n$ with $(k-1)$ dim boundary oriented consistent with S.

ω is a $(k-1)$ form
then

$$\int_{\partial S} \omega = \int_S d\omega$$