Feb 16

Note Title 2/16/2009

p. 266 #3, 4

S - unit sphere in $\mathbb{R}^3$ -

$$\iint_S (x^2+y^2-2z^2)\, dA = 0$$

$$= \iint_S (x^2 - z^2) + (y^2 - z^2) \, dA$$

$$\iint_S \underline{\underline{(x^2 - z^2)}} \, dA$$

interchange $\underline{x \leftrightarrow z}$

Mirror refl.

takes $S \to S$.

preserves integrals

but $x^2 - z^2 \to z^2 - x^2$.

so by symm

$$\iint_S (x^2 - z^2)\, dA = 0$$

$$\iint_S (y^2 - z^2)\, dA = 0$$

$$\operatorname{div}(\operatorname{grad} F \times \operatorname{grad} G) = 0$$

$$\operatorname{div}(F \times G)$$

$$= G \cdot \operatorname{curl} F - F \cdot \operatorname{curl} G$$

$$\operatorname{grad} G \cdot \overset{0}{\cancel{\operatorname{curl}\operatorname{grad} F}}$$

$$- \operatorname{grad} F \cdot \overset{0}{\cancel{\operatorname{curl}\operatorname{grad} G}}$$

Last time $R \subseteq \mathbb{R}^n$

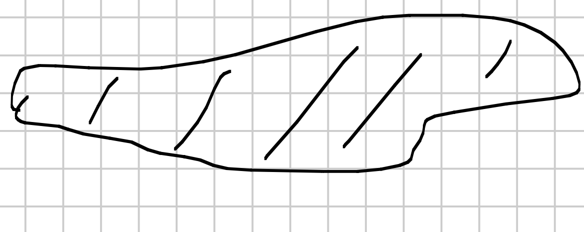
simply connected.

For any curve — C

if $\partial C = \emptyset$

then C is $\underline{\partial S}$

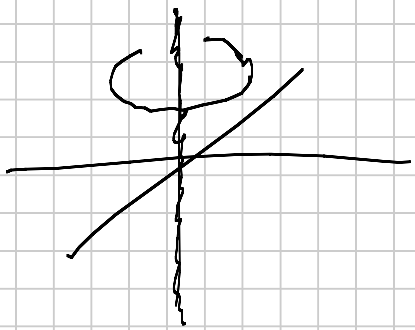
in R.



in this case a
vector field G
is a grad(f)
iff all

$$\frac{\partial G_i}{\partial x_j} = \frac{\partial G_j}{\partial x_i}$$

Ex. $\mathbb{R}^3$ - remove z-axis -
what remains is R



$$G(x,y,z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$$

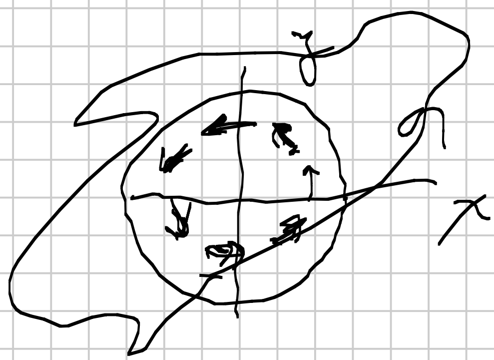
curl G

$$\begin{bmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{bmatrix}$$

$$= \left(0,\ 0,\ \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) \right)$$

$$= \frac{(x^2+y^2) - x(2x) + (x^2+y^2) - y(2y)}{(x^2+y^2)^2}$$

$$= 0$$



$$G = \left(\frac{-\sin\theta}{r},\ \frac{\cos\theta}{r} \right)$$

C - circle around origin at radius r

$$\int_C G \cdot dx = \underline{\underline{2\pi}}$$

$\underline{\underline{\text{so}}}$ G cannot be a $\underline{\text{gradient}}$

Verge of a beautiful & simple idea ~

$\mathbb{R}^3$

$$\underset{\text{fctn}}{\times} \xrightarrow{\text{grad}} \underset{\text{v.f.}}{\times} \xrightarrow{\text{curl}} \underset{\text{v.f.}}{\times} \xrightarrow{\text{div}} \underset{\text{fctn}}{\times}$$

if $\boxed{\text{Simply connected}}$ then G is in image of grad, iff curl $G = 0$.

$\mathbb{R}^2$

$$\underset{\text{fctn}}{\times} \xrightarrow{\text{grad}} \underset{\text{v f}}{\times} \xrightarrow{\frac{\partial G_1}{\partial x_2} - \frac{\partial G_2}{\partial x_1}} \underset{\text{fctn}}{\times}$$

$\mathbb{R}^n$

$X \longrightarrow X \longrightarrow X$

have element in middle comes from left iff it goes to 0.

$X \xrightarrow{grd} X \longrightarrow \; ? \; ?$

fctn v.f.

$R \subseteq \mathbb{R}^n$ - Convex

if $\vec{x}, \vec{y} \in C$

then $\alpha \vec{x} + (1-\alpha)\vec{y} \in C$

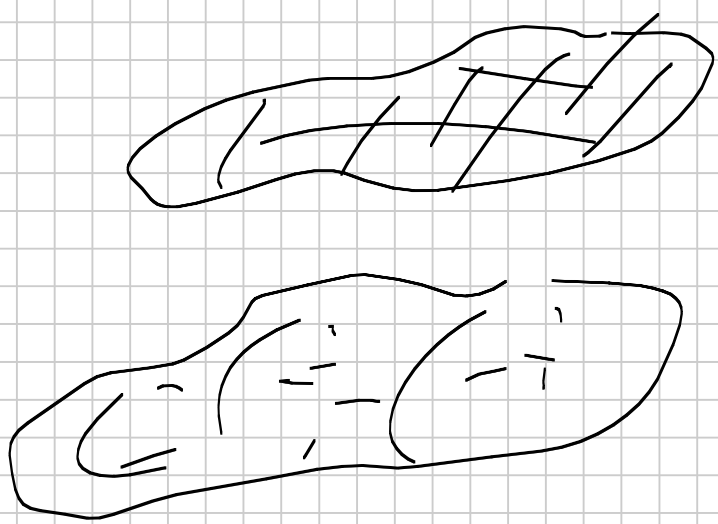
$\alpha \in [0, 1]$.

Means something —

If S is a k-dim hypersurface, C^1.

$0 \le k \le n$

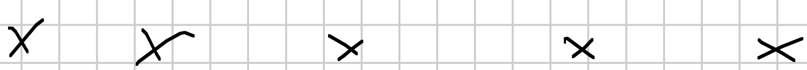
with no $k-1$ dim boundary, then S is the boundary of some $k+1$ dim hypersurface.



i.e. $\partial S = \emptyset$

$\iff S = \partial M.$

Orienting



Things that integrate over hypersurfaces

$$\int_C \underline{\underline{G}} \cdot \underline{\underline{dx}}$$

Grassman $\underline{1840\text{'s}}$

Grassman Calculus
Exterior Calculus

<u>fctns</u> – "0 forms"

$+x_1$ $\quad -x_2$ $\quad +x_3$ $\quad +x_4$

$$f(x_1) - f(x_2) + f(x_3) + f(x_4).$$

grad

$$\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \cdots \frac{\partial f}{\partial x_n}\right)$$

$$\frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 \cdots \frac{\partial f}{\partial x_n} dx_n$$

basis elements

1 - forms

0 forms $\quad$ 1 forms $\quad$ 2 forms $\quad$ 3 forms $\quad \cdots \quad$ n forms

<u>basis</u> <u>2 forms</u>, $\mathbb{R}^3$

$dx_1 \underset{=}{\wedge} dx_2$, $dx_1 \underset{=}{\wedge} dx_3$, $dx_3 \wedge dx_1$

"wedge"

Grassman -

If α & β are forms

so is $\alpha \wedge \beta$

& $\alpha \wedge \beta = -\beta \wedge \alpha$.