Apr 10

Note Title 4/10/2009

Fourier series

1) $\sum_{n=-\infty}^{\infty} c_n e^{inx}$

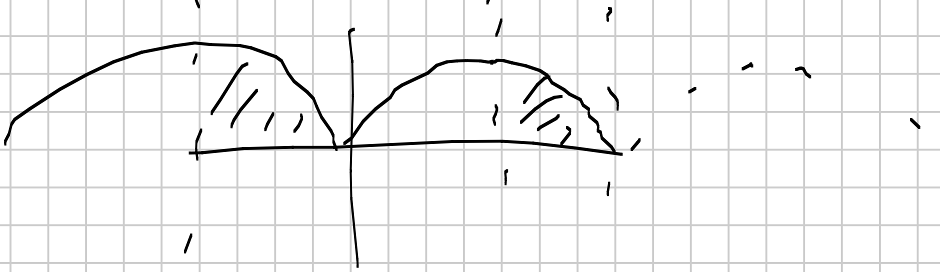
2) $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$\left.\begin{aligned} a_n &= c_n + c_{-n} \\ b_n &= c_n - c_{-n} \end{aligned}\right\}$$

$$c_n = \int_0^{2\pi} f(x) e^{-inx} \, dx$$

$$= \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$$

f – 2π-per.



"Orthogonality"

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\overline{g}(x)\,dx$$

All e^{inx} are <u>orth.</u>

$$\langle e^{inx}, e^{inx} \rangle = \| e^{inx} \|_2^2$$

$$= \underline{2\pi}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

$\cos nx \qquad \sin nx$

These are also orthogonal

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\cos nx = \frac{e^{inx} + e^{-inx}}{2}$$

$$\sin nx = \frac{e^{inx} - e^{-inx}}{2i}$$

$$\langle \cos nx, \cos mx \rangle$$

$$= \int_{-\pi}^{\pi} \left(\frac{e^{inx} + e^{-inx}}{2} \right) \left(\frac{e^{-imx} + e^{imx}}{2} \right) dx$$

$$= \frac{1}{4} \int_{-\pi}^{\pi} \left(e^{i(n-m)x} + e^{i(n+m)x} + e^{-i(n+m)x} + e^{-i(n-m)x} \right) dx$$

$$n, m \geq 0$$

$$= \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases}$$

Same is true

$$\langle \sin nx, \sin mx \rangle$$

$$= \begin{cases} 0, & n \neq m \\ \pi, & n = m \end{cases}$$

$$\langle \cos nx, \sin mx \rangle$$

$$= \int_{-\pi}^{\pi} \left(\frac{e^{inx} + e^{-inx}}{2}\right) \left(\frac{e^{-imx} - e^{+imx}}{-2i}\right) dx$$

$$= \frac{1}{4i} \int_{-\pi}^{\pi} \left(e^{i(n-m)x} - e^{i(n+m)x} + e^{-i(n+m)x} - e^{-i(n-m)x}\right) dx$$

$$\begin{cases} 0 & n \neq m \\ 0 & n = m \end{cases}$$

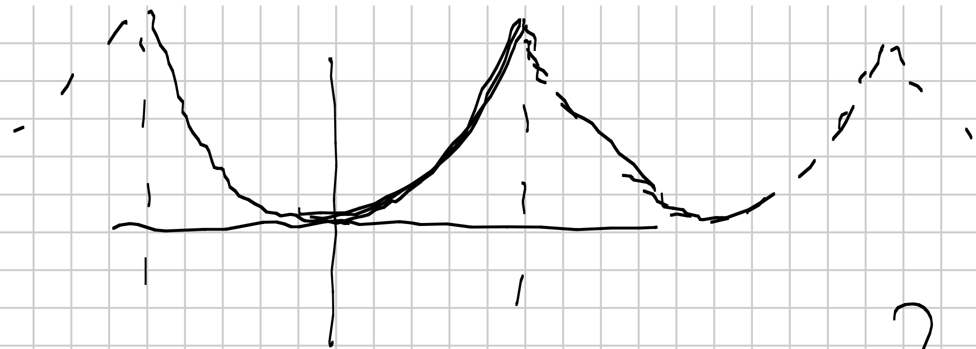
$\cos nx, \quad \sin nx$ — all orth.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad n \geq 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad n \geq 1$$

Ex

$$f(x) = x^2 \quad -\pi \leq x \leq \pi$$



What are c_n, a_n, b_n?

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}\,dx$$

$$n = 0, \quad c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{1}{2\pi}\,\frac{x^3}{3}\Big|_{-\pi}^{\pi} = \frac{2\pi^3}{6\pi} = \boxed{\frac{\pi^2}{3}}$$

$$\int x^2 e^{-inx} = -\frac{x^2 e^{-inx}}{in} + \frac{2}{in}\underbrace{\int x e^{-inx}\,dx}$$

$u = x^2$

$dv = e^{-inx}\,dx$

$v = \dfrac{e^{-inx}}{-in}$

$du = 2x\,dx$

$$\int x e^{-inx}\,dx = -\frac{x e^{-inx}}{in} + \frac{1}{in}\underbrace{\int e^{-inx}\,dx}$$

$u = x$

$dv = e^{-inx}\,dx$

$du = dx$

$v = \dfrac{e^{-inx}}{-in}$

$$\int e^{-inx}\,dx = -\frac{1}{in}e^{-inx}$$

$$\int x^2 e^{-inx} = -\frac{x^2 e^{-inx}}{in} + \frac{2xe^{-inx}}{n^2} - \frac{2}{(in)^3}e^{-inx}$$

$$= \left(-\frac{x^2}{in} + \frac{2x}{n^2} + \frac{2}{in^3}\right)e^{-inx}$$

$$C_n = \frac{1}{2\pi}\left(-\frac{x^2}{in} + \frac{2x}{n^2} + \frac{2}{in^3}\right) e^{-inx}\Bigg|_{-\pi}^{\pi}$$

at $-\pi$ & π,

$$e^{-inx} = (-1)^n$$

$$e^{\pm in\pi} = (e^{\pm i\pi})^n$$

$$= \frac{1}{2\pi}\left(\frac{-x^2}{in} + \frac{2x}{n^2} + \frac{2}{in^3}\right)(-1)^n\Bigg|_{-\pi}^{\pi}$$

cancel

$$\frac{2}{n^2}(-1)^n = \frac{\cancel{2}}{\cancel{2\pi}}\left(\frac{2\cancel{\pi}}{n^2}\right)(-1)^n = C_{-n}$$

<u>Note</u> <u>all</u> coeff are real #'s

$$\& \quad C_{-n} = C_n$$

$$\& \quad C_0 = \pi^2/3$$

$$a_n = C_n + C_{-n} = 2C_n = \frac{4}{n^2}(-1)^n$$

$$b_n = \underline{C_n - C_{-n}} = 0$$

$$f(x) = \pi^2/3 + \underbrace{\sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos(nx)}$$

alt series, $\frac{4}{n^2}\cos(nx)$

decrease?

so this series
conv. absol.
& uniformly

? Does it actually
conv. to the
original f?