M417, Fall 2008, Second hourly exam

This is a closed book, closed notes exam. Show all your work. For full credit you must show complete arguments.

Prob. 1 (30 pts)
Find the extreme values of $f(x, y) = 3x^2 - 2y^2 + 2y$ on the set $S = \{(x, y) : x^2 + y^2 \leq 1\}$.

Prob. 2 (35 pts)

a) Show that the pair of equations $xy + 2y^2 - 3xz = 0$ and $xyz + x - \overset{2y}{\cancel{y}} = 0$ can be solved for x and y as C^1 functions of z near the point $(1, 1, 1)$.

b) If we let $x = f(z)$ and $y = g(z)$ be these solutions what are $f'(1)$ and $g'(1)$?

Prob. 3 (35 pts)
Suppose $f : [a, b] \to \mathbb{R}$ is continuous and differentiable on (a, b). Moreover suppose there is a bound B and for all $x \in (a, b)$, $|f'(x)| < B$.

Recall that for any partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$,

$$\text{mesh}(P) = \max((x_i - x_{i-1}), i = 1, \ldots, n).$$

Prove that for all partitions P of $[a, b]$ that

$$\overline{S}_P(f) - \underline{S}_P(f) \leq B(b - a)\text{mesh}(P).$$

$$\overline{S}_P(f) - \underline{S}_P(f) = \sum_{i=1}^{n} (M_i - m_i)\, \ell(I_i)$$

$$\leq B \,\text{mesh}\, P \cdot \sum_{i=1}^{n} \ell(I_i)$$

$$= \sum_{i=1}^{n} (f(a_i) - f(b_i))\, \ell(I_i)$$

MVT $\quad |f(a_i) - f(b_i)|$

$$= |f'(c_i)(a_i - b_i)|$$

$$\leq B\, \ell(I_i)$$

$$\overline{S}_P(f) - \underline{S}_P(f) \leq B \sum_{i=1}^{n} \ell(I_i) \cdot \ell(I_i)$$

$$\leq B \,\text{mesh}\, P\, (b - a)$$

$3x^2 - 2y^2 + 2y$

$(x^2 + y^2) \leq 1$

Consider

$x^2 + y^2 < 1$

Find crit. pts.

$\partial_x f = 6x = 0$

$\partial_y f = -4y + 2 = 0$

$x = 0$ $(0, \frac{1}{2})$

$y = \frac{1}{2}$

$x^2 + y^2 = 1$

$\gamma(t) = (\cos t, \sin t)$

$g(t) = f(\gamma(t))$

$= 3\cos^2 t - 2\sin^2 t + 2\sin t$

$g'(t) = -6\cos t \sin t - 4\cos t \sin t$

$+ 2\cos t$

$2\cos t - 10\cos t \sin t = 0$

$\cos t(2 - 10\sin t) = 0$

$\cos t = 0$

$\sin t = \frac{2}{10}$

$(0, 1), (0, -1)$

$\left(\frac{4\sqrt{6}}{10}, \frac{2}{10}\right), \left(\frac{4\sqrt{6}}{10}, \frac{2}{10}\right), (0, \frac{1}{2})$

$x = \pm\sqrt{1 - \frac{4}{100}} = \frac{4\sqrt{6}}{10}$

Eval. 5 values

& take max & min

$$xy + 2y^2 - 3xz = 0$$

$$xyz + x - 2y = 0$$

x & y as fctns of z.

$G(x, y, z)$, at $(1, 1, 1)$

$$\left[\begin{array}{cc|c} -2 & 5 & -3 \\ 2 & -1 & 1 \end{array}\right] \qquad B = \begin{bmatrix} -2 & 5 \\ 2 & -1 \end{bmatrix}, \quad A = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

need nonzero det.

it is -8

$$\begin{bmatrix} -2 & 5 \\ 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & -5 \\ -2 & -2 \end{bmatrix} \Big/ -8 = \begin{bmatrix} 1/8 & 5/8 \\ 1/4 & 1/4 \end{bmatrix}$$

∴ IFT guarantees in a nghd of $(1, 1, 1)$ (x, y) can be found as C^1-fctns. of z.

Let them be $f(z), g(z)$

$$\begin{pmatrix} f'(1) \\ g'(1) \end{pmatrix} = -B^{-1}A$$

$$\begin{pmatrix} f'(1) \\ g'(1) \end{pmatrix} = - \begin{bmatrix} 1/8 & 5/8 \\ 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$= \begin{pmatrix} -1/4 \\ 1/2 \end{pmatrix} \checkmark$$