# M417, Fall 2008, First hourly exam

This is a closed book, closed notes exam. Show all your work. For full credit you must show complete arguments.

**Prob. 1** (30 pts)
Define a sequence $\{x_k\}$ recursively by $x_1 = \sqrt{2}$, $x_{k+1} = \sqrt{2 + x_k}$. Show by induction that (a) $x_k < 2$ and (b) $x_k < x_{k+1}$ for all $k$. Then show that $\lim x_k$ exists and evaluate it.

**Prob. 2** (35 pts)
For $S \subseteq \mathbb{R}^n$ we define the *diameter* of $S$ to be

$$\text{dia}(S) = \sup(|x - y| : x, y \in S).$$

Notice that this value could be $\infty$.

a. Prove that if $\text{dia}(S) < \infty$ then $S$ is bounded.

b. Prove that if $K \subseteq \mathbb{R}^n$ is compact and nonempty then there must be points $x_0, y_0 \in K$ for which

$$\text{dia}(K) = |x_0 - y_0|.$$

**Prob. 3** (35 pts)
The set $S = \{(x, y, z) : 2x^2 + y^2 + z^2 - 4 = 0\}$ is an ellipsoid centered at the origin.

a. Find the two points where the curve $\gamma(t) = (t^2, t, t)$ intersects this ellipsoid.

b. Show that at these two points the tangent vector to the curve is orthogonal to the tangent plane to the ellipsoid.