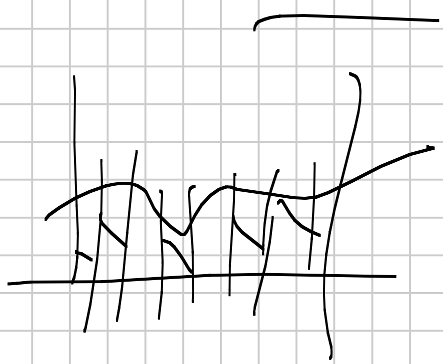
Review

Note Title 12/14/2009

$f: [a,b]$, $\underline{f(x) \geq 0}$

$S = \{(x,y) : 0 \leq y \leq f(x),$
$x \in [a,b]\}$.



$$P_n = \{a, a+\frac{b-a}{n}, a+\frac{2(b-a)}{n} \ldots b\}$$

$$\overline{S}(f, P_n) \searrow \int_a^b f$$
$$\underline{S}(f, P_n) \nearrow$$

$$\overline{S}(f, P_n) = \sum_{i=1}^{n} M_i \, \ell(I_i)$$
$$\underline{S}(f, P_n) = \sum_{i=1}^{n} m_i \, \ell(I_i).$$

Rectangles

$M_i \, l(I_i)$

$R_i = [0, M_i] \times I_i$

$$S \subseteq \bigcup_{i=1}^{n} R_i$$

$$\sum_{i=1}^{n} \text{Area}(R_i) = U(f, P_n)$$

$r_i = [0, m_i] \times I_i$

etc

S is inf.

So I can select $\{x_k\}$, $x_k \in S$ & all different.

pick x_1 —

$S - \{x_1\}$.

pick x_2 etc

$p(i) = x_i$

is 1-1.

x_i - bdd

$\Rightarrow \exists$ conv. subs.

$x_{k_i} \to x$

Show x is an a.pt.

in $B_r(x)$, $r > 0$.

$\exists N$, $\forall n \geq N$

$|x_{k_n} - x| < r$.

i.e. $p(\{k_N, k_{N+1}, k_{N+2} \dots\})$

$\subseteq B_r(x)$.

p is 1-1

&

$\{k_N, k_{N+1}, k_{N+2} \dots\}$

is inf.

so $p(\{ \quad \})$ is inf.

$$F : (\vec{x}, \vec{y})$$

$$\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$$

$$DF(\vec{a},\vec{b}) = \left[\begin{array}{c|c} A & B \end{array}\right] \qquad F(\vec{x},\vec{y}) = \cancel{0} \; F(\vec{a},\vec{b})$$

$$\det(B) \neq 0$$

$$G(\vec{x},\vec{y}) = (\vec{x}, F(\vec{x},\vec{y}))$$

$$\mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$$

$$DG(\vec{a},\vec{b}) = \begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} \vec{x} \\ F \end{array} & \left[\begin{array}{c|c} I & 0 \\ \hline A & B \end{array}\right] \end{array}$$

$$\det(DG(\vec{a},\vec{b})) = \det(B) \neq 0.$$

I.F.T.

in a nghd of
$G(\vec{a},\vec{b})$, G has
a C^1-inverse to
a nghd of $(\vec{a},\vec{b})$.

Let V be nghd
of $G(\vec{a},\vec{b})$

$$V_1 = \{(\vec{x}, F(\vec{a},\vec{b})) \in V\}$$

nghd of $(\vec{a}, F(\vec{a},\vec{b}))$

$$G(\vec{x},\vec{y}) = (\vec{x}, F(\vec{x},\vec{y}))$$

$$G^{-1}(\vec{x},\vec{z}) = (\vec{x}, \underline{\underline{g(\vec{x},\vec{z})}})$$

on V_1,

$$G^{-1}(\vec{x}, F(\vec{a},\vec{b}))$$

$$= (\vec{x}, \underbrace{g(\vec{x}, F(a,b))}_{f(\vec{x})}).$$

$$G(G^{-1}(\vec{x}, F(\vec{a},\vec{b})))$$

$$= (\vec{x}, F(\vec{a},\vec{b}))$$

$$= G(\vec{x}, f(\vec{x})) = (\vec{x}, F(\vec{x}, f(\vec{x})))$$

$$\Rightarrow F(\vec{a}, \vec{b})$$
$$= \underline{\underline{F(\vec{x}, f(\vec{x}))}} .$$

$$F(\vec{x}, f(\vec{x})) = F(\underline{\underline{a, b}})$$
$$\underset{u(\vec{x})}{\shortparallel}$$

$$D(u) = [0]$$

$$D_{\vec{x}}(F(\vec{x}, f(\vec{x})) \quad \mathbb{R}^n \to \mathbb{R}^m$$

$$= \begin{bmatrix} \; \end{bmatrix}$$

$$= D_{\vec{x}}(F)(\vec{x}, f(\vec{x}))$$
$$+$$
$$D_{\vec{y}}(F)(\vec{x}, f(\vec{x})) D_{\vec{x}}(f)(\vec{x}) = 0$$

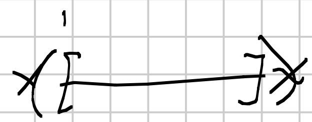
$$\text{at } (\vec{a}, \vec{b})$$

$$D_{\vec{x}}(F)(\vec{a}, \vec{b}) +$$
$$D_{\vec{y}}(F)(\underset{\vec{x}}{\vec{a}}, \underset{\vec{y}}{\vec{b}}) D_{\vec{x}}(f)(\vec{a}) = 0$$

$$DF(\vec{a}, \vec{b}) \quad \left[A \;\vdots\; B \right]$$

$$A + B D_{\vec{x}}(f)(\vec{a}) = 0$$

$$D_{\vec{x}}(f)(\vec{a}) = -B^{-1}A$$

 f - c(a)

$0 \qquad 1$

$[\frac{1}{n}, 1-\frac{1}{n}]$

f - cont on $(0, 1]$

$\lim_{x \to 0^+} f(x) = L$.

Suppose $f: (0,1) \to \mathbb{R}$

• if x_k is Cauchy

the $f(x_k)$ is Cauchy

is f. unif. cont?

is f cont?

Suppose $x_k \to x$

does $f(x_k) \to f(x)$?

$$f(x) = \begin{cases} 1, & x = 1/2 \\ 0 & \text{otherwise} \end{cases}$$

$$x_k = \frac{1}{2} - \frac{1}{k}.$$

$$f(x_k) = 0, \quad x_k \to \frac{1}{2}.$$

$$x_1 = \sqrt{2}$$

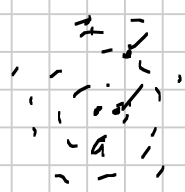
$$\boxed{x_{k+1} = \sqrt{2 + x_k}}$$

$$\lim_{k \to \infty} x_k = a$$

$$a = \sqrt{2 + a}$$

$$a^2 = 2 + a, \quad a^2 - a - 2 = 0$$

$$a = \frac{1 + \sqrt{1 + 4}}{2}$$



a - <u>not</u> an accum pt

$\Rightarrow \exists\, r > 0,$

$B_r(a) \cap S$ is finite

$= \{s_1, s_2 \dots s_k\}$

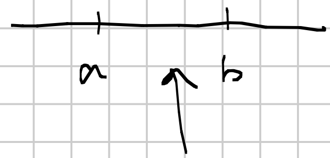
$\min(\|a - s_i\| : i = 1 \dots k,$
$s_i \neq a) = r > 0$

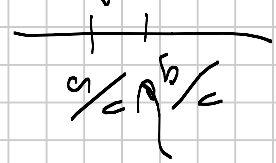
$B_r(a) \cap S$ has no elts of S except perhaps a.

$f : [a,b] \to \mathbb{R}$
R. int.

$$\int_a^b f(x)dx = c \int_{a/c}^{b/c} f(cx)dx$$

$f(x)$ $\qquad$ $g(x) = f(cx)$





$P = (x_1 \dots x_n)$, $\quad Q = (x_1/c, \dots x_n/c)$

$I_i = [x_{i-1}, x_i] \longleftrightarrow J_i = [\frac{x_{i-1}}{c}, \frac{x_i}{c}]$

$\sup(f(x) : x \in I_i) = \sup(g(x) : x \in J_i)$.

$x_{i-1} \le x \le x_i$ $\qquad$ $\frac{x_{i-1}}{c} \le x \le \frac{x_i}{c}$

$f(x)$ $\qquad$ $g(x) = f(cx)$

$x_{i-1} \le cx \le x_i$

$$\ell(I_i) = x_i - x_{i-1}$$

$$\ell(J_i) = x_i/c - x_{i-1}/c$$

$$= \ell(I_i)/c.$$

$$\overline{S}(f,P) = \sum_{i=1}^{n} M_i \, \ell(I_i)$$

$$= c\sum_{i=1}^{n} M_i \, \ell(J_i) = c\,\overline{S}(g,Q).$$

$\underline{S}(f,P) = c\,\underline{S}(g,Q)$ - same reasoning

$S \subseteq \mathbb{R}^n$

Bounded

$|x-a| < r$

$(a-r, a+r)$

$\mathbb{R}^n$, $\quad B_r(a)$

$$= \{\vec{x} : \|\vec{x}-\vec{a}\| < r\},$$

$S \neq \emptyset$

$\exists\, x_1 \in S$

$S - \{x_1\} \neq \emptyset$

$\exists\, x_2 \neq x_1$, in S

$S - \{x_1, x_2\} \neq \emptyset$

x_3

Get $x_1, x_2, x_3 \ldots$

all x_i's are different

$p : \mathbb{N} \to S$

$p(i) = x_i$

is 1-1

$x_i \in S$

so Bounded seq. in $\mathbb{R}^n$

so $\exists$ conv. subseqn.

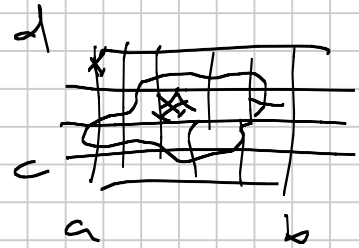
$x_{k_i} \to x$

S

inner area -

$S \subseteq [a,b]\times[c,d]$.

$P\times Q$



Rectangles $\subseteq S$

Sum there areas.

$$\underline{A}(S, P\times Q) = \underline{S}(\chi_S, P\times Q)$$

Rectangles

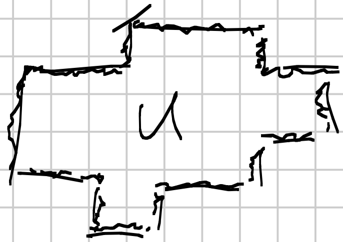
$S^{int} \subseteq S$

inner area $S^{int} \leq$ inner area S.

$\overline{P\times Q}$

Take the rect that are $\subseteq S$.

$R_1, R_2 \dots R_s$



Shave off the edge to get $\underline{\mathcal{U}}$ -

$\mathcal{U} \subseteq \bar{S}^{int}$

& Area $\mathcal{U} >$

$\underline{A}(S, P \times Q) - \varepsilon'$

$\Rightarrow \underline{A}(S, P \times Q) - \varepsilon$

$\wedge$

Area $\mathcal{U}$

$\wedge$

$\bar{\text{inner}}$ area of S^{int}.

Let $\varepsilon \to 0$

$\underline{A}(S, P \times Q) \leq$ inner area of S^{int}

Let $P \times Q$ refine

inner area $S \leq$ inner area S^{int}

x_k - conv.

$\forall \varepsilon > 0$ $\exists$ int.

$I_1, \dots, I_k$

$S \subseteq \bigcup I_i$

$\sum \ell(I_i) < \varepsilon$

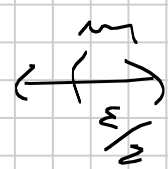
$S = \{ x_k \}$.

$x_k = \frac{1}{k}$

$\cancel{(-\delta, \delta)}$

$\cancel{(0, \varepsilon)}$

$(-\frac{\varepsilon}{4}, \frac{\varepsilon}{4}) = I_1$



fin many pts left.

$x_1, \dots, x_k$.

$(x_i - \frac{\varepsilon}{4k}, x_i + \frac{\varepsilon}{4k}) = I_{i+1}$

$$x_k \to a$$

$$I_1 = \left(a - \frac{\varepsilon}{4}, a + \frac{\varepsilon}{4}\right)$$

$\exists\, K,\ k \geq K,$

$$|x_k - a| < \varepsilon/4.$$

so $k \geq K,$

$$x_k \in I_1,$$

$x_1, \dots, x_{K-1}$

$$I_{i+1} = \left(x_i - \frac{\varepsilon}{4K}, x_i + \frac{\varepsilon}{4K}\right)$$

1) if $i \leq K,$

$x_i \in I_1$

if $i > K,$

$x_i \in I_{i+1}$

so $\{x_i\} \subseteq \bigcup_{i=1}^{K} I_i.$

2) $\sum_{i=1}^{K} l(I_i)$

$$= l(I_1) + (K-1)\frac{\varepsilon}{2K}$$

$$= \frac{\varepsilon}{2} + \underbrace{\frac{K-1}{K}}_{<1}\varepsilon/2$$

$< \varepsilon$.

f: R. int.

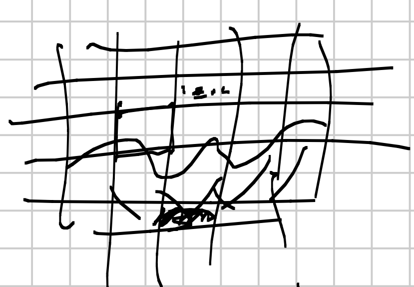
$f \geq 0$.

$S = \{(x,y) : 0 \leq y \leq f(x) : x \in [a,b]\}$

S is Jordan measurable.

Inner & outer areas

$P \times Q$



(Rect. $\subseteq S$) - add up areas
- inner area

(Rect that int. S)
- add up areas
- outer area.

$\overline{S}(\chi_S, P \times Q)$

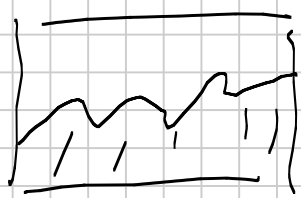
$=$ outer sum.

$\underline{S}(\chi_S, P \times Q)$

$=$ inner sum.

S is msble $\Longleftrightarrow$

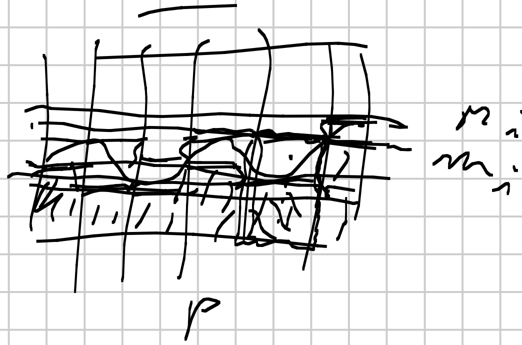
χ_S is integrable



$\forall \varepsilon > 0$

$\exists P$

$\overline{S}(f, P) - \underline{S}(f, P) < \varepsilon.$



$P = \{x_0, \dots x_n\}$

$\{M_i, m_i\}$

$$= \{y_0 < y_1 < y_2 < \cdots y_{m-1} < y_m = M\}$$

$$\parallel$$

$$Q$$

$$\overline{S}(\chi_S, P \times Q)$$
$$= \overline{S}(f, P)$$

$$\underline{S}(\chi_S, P \times Q)$$
$$= \underline{S}(f, P)$$

χ_S is R-int.

iff. f is R. int.

$$\overline{S}(f, P) = \overline{S}(\chi_S, P \times Q)$$
$$\geq \underline{S}(\chi_S, P \times Q) = \underline{S}(f, P)$$

Choose P_i so that
mesh$(P_i) \to 0$.

$$\overline{S}(f, P_i) + \underline{S}(f, P_i) \to \int f$$

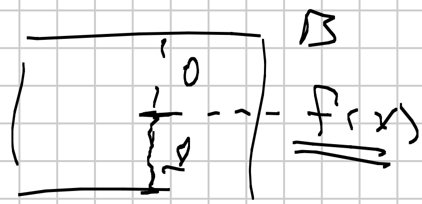
$$\overline{S}(\chi_S, P_i \times Q_i) \longrightarrow \int f$$

$$\underline{S}(\chi_S, P_i \times Q_i) \nearrow$$

$$\Rightarrow \overline{I}(\chi_S) = \underline{I}(\chi_S) = \iint \chi_S = \int f$$

$$\int \chi_S \, dA$$

$$= \int_a^b \underbrace{\int_0^B \chi_S(x,y)\, dy}\, dx$$



$$= \int_a^b f(x)\, dx$$