Sept 30

Note Title 9/30/2009

If f has partials near $\vec{a}$ & cont at $\vec{a}$ then f is diff at $\vec{a}$.

Suppose D - open & $f: D \to \mathbb{R}$ & has cont. partial derivs on D - Then we say f is "of class C^1 on D", $f \in C^1(D)$

Chain Rule

$$f : \mathbb{R}^n \to \mathbb{R}^m$$

$$f = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}$$

$$f_i : \mathbb{R}^n \to \mathbb{R}$$

assume diff at $\vec{a}$.

$$\begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix} = \begin{pmatrix} f_1(\vec{a}) \\ f_2(\vec{a}) \\ \vdots \\ f_m(\vec{a}) \end{pmatrix} + \underbrace{\begin{bmatrix} \nabla f_1(\vec{a}) \\ \nabla f_2(\vec{a}) \\ \vdots \\ \nabla f_m(\vec{a}) \end{bmatrix}}\begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ \vdots \\ x_n - a_n \end{pmatrix} + \begin{pmatrix} E_1(\vec{x}) \\ E_2(\vec{x}) \\ \vdots \\ E_m(x) \end{pmatrix}$$

$$f(\vec{x}) = f(\vec{a}) + [Df](\vec{x}-\vec{a}) + E(x)$$

$$\lim_{x \to a} \frac{\| E(\vec{x}) \|}{\| \vec{x} - \vec{a} \|} = 0$$

$$[Df(\vec{a})] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{a}), & \frac{\partial f_1}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{a}) \\ \frac{\partial f_2}{\partial x_1}(\vec{a}), & \frac{\partial f_2}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial f_2}{\partial x_n}(\vec{a}) \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1}(\vec{a}), & \frac{\partial f_m}{\partial x_2}(\vec{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\vec{a}) \end{bmatrix}$$

Chain Rule

$f : \mathbb{R}^n \to \mathbb{R}^m$

$g : \mathbb{R}^m \to \mathbb{R}^k$

f is diff at $\vec{a}$

g is diff at $f(\vec{a})$.

$$f(x) = f(\vec{a}) + M(\vec{x} - \vec{a}) + E_f(\vec{x})$$

$$g(f(x)) = g(f(\vec{a})) + N(\underline{f(\vec{x}) - f(\vec{a})}) + E_g(f(x))$$

$$= g(f(\vec{a})) + N(M(\vec{x} - \vec{a}) + E_f(\vec{x})) + E_g(f(x)).$$

$$= g(f(\vec{a})) + (NM)(\vec{x} - \vec{a}) + \underbrace{N(E_f(x)) + E_g(f(x))}_{} = E(x)$$

Want to show

$$\lim_{x \to \vec{a}} \frac{\|E(x)\|}{\|\vec{x} - \vec{a}\|} = 0.$$

$$\frac{\|N(E_f(x))\|}{\|\vec{x} - \vec{a}\|}$$

$\exists$ K so that

$$\|N\vec{v}\| \leq K\|\vec{v}\|$$

$$\frac{\|N(E_f(x))\|}{\|\vec{x}-\vec{a}\|} \leq K\frac{\|E_f(x)\|}{\|\vec{x}-\vec{a}\|} \to 0$$

2nd term

$$\frac{\|E_g(f(x))\|}{\|\vec{x}-\vec{a}\|}$$

$$= \frac{\|E_g(f(\vec{x}))\|}{\|f(\vec{x})-f(\vec{a})\|} \cdot \left(\frac{\|f(\vec{x})-f(\vec{a})\|}{\|\vec{x}-\vec{a}\|} \right)$$

so long as $f(\vec{x}) \neq f(\vec{a})$

$= 0$ if it is.

& I'm done.

$$\frac{\|f(\vec{x})-f(\vec{a})\|}{\|\vec{x}-\vec{a}\|} \leq \frac{\|M(\vec{x}-\vec{a})\| + \|E_f(\vec{x})\|}{\|\vec{x}-\vec{a}\|} \to 0$$

$$\frac{\|M(\vec{x}-\vec{a})\|}{\|\vec{x}-\vec{a}\|} \text{ is bounded}$$

by some B.

$$\lim_{\vec{x}\to\vec{a}} \frac{\|E_g(f(\vec{x}))\|}{\|f(\vec{x})-f(\vec{a})\|} = \lim_{f(x)\to f(a)} \frac{\|E_g(f(x))\|}{\|f(x)-f(a)\|}$$

$$= 0$$

So 2nd term $\to 0$

Conclusion –

$$\overset{k\times n}{D(g\circ f)(\vec{a})} = \overset{k\times m}{\underline{Dg}(f(\vec{a}))}\,\overset{m\times n}{\underline{Df}(\vec{a})}$$

<u>Ex</u> $f: \mathbb{R}^n \to \mathbb{R}$

+ each x_i, $i = 1 \cdots n$

is a fctn of t,

$$x_i(t)$$

$$\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} : \mathbb{R} \to \mathbb{R}^n$$

Let

$$w(t) = f(x_1(t), \ldots, x_n(t))$$

What is $\dfrac{dw}{dt}$?

$$\begin{array}{ccccc} t & \xrightarrow{\vec{x}} & \vec{x} & \xrightarrow{f} & \cdot \\ \mathbb{R} & & \mathbb{R}^n & & \mathbb{R} \end{array}$$

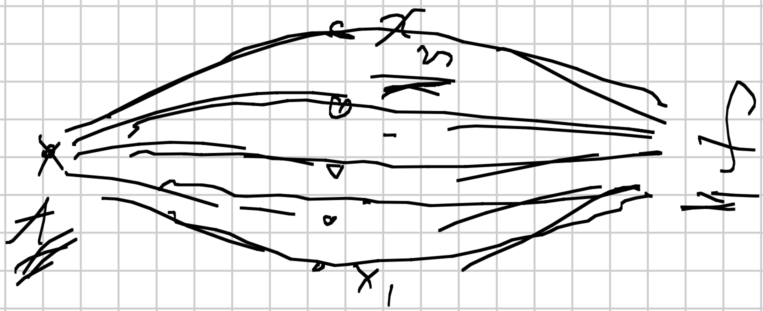
$$w(t) = f(\vec{x}(t))$$

$$\frac{dw}{dt} = Df(\vec{x}(t))\, D(\vec{x})(t)$$

$$\left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \cdots \frac{\partial f}{\partial x_n}\right) \begin{pmatrix} \frac{dx_1}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{pmatrix}$$

$$= \frac{\partial f}{\partial x_1}(x) \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dt}$$

$f \cdots \frac{\partial f}{\partial x_n} \frac{dx_n}{dt}$



Ex f is diff fct of (x, y)

$$x = s \log(1+t^2)$$

$$y = \cos(s^3 + st)$$

$$v(s,t) = f(s\log(1+t^2), \cos(s^3+st))$$

$$\begin{tikzcd} \overset{(s,t)}{\mathbb{R}^2} \arrow[r] \arrow[rr, bend right, "v"'] & \overset{(x,y)}{\mathbb{R}^2} \arrow[r, "f"] & \mathbb{R}. \end{tikzcd}$$

what are $\frac{\partial v}{\partial t}$, $\frac{\partial v}{\partial s}$?

$$\frac{\partial V}{\partial t} = \frac{\partial f}{\partial x}\left(\frac{2st}{1+t^2}\right) - \frac{\partial f}{\partial y}\left(s\sin(s^3 + st)\right)$$