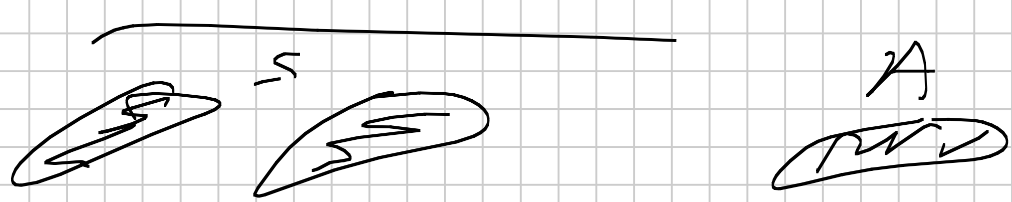
Sept 23

Note Title 9/23/2009

Connected

Lemma Suppose $S \subseteq \mathbb{R}^n$ & is disconnected by S_1 S_2. & $A \subseteq \mathbb{R}^m$ is connected & $f : A \to S$ is continuous. Then either $f(A) \subseteq S_1$ or $f(A) \subseteq S_2$.



pf. $f^{-1}(S_1) = A_1$

$f^{-1}(S_2) = A_2$

1) $A_1 \cup A_2 = A$

2) $$\overline{A_1} \cap A_2 = \overline{f^{-1}(S_1)} \cap f^{-1}(S_2)$$
$$\subseteq f^{-1}(\overline{S_1}) \cap f^{-1}(S_2)$$
$$= f^{-1}(\overline{S_1} \cap S_2) = \emptyset.$$

$A_1 \cap \overline{A_2} = \emptyset$.

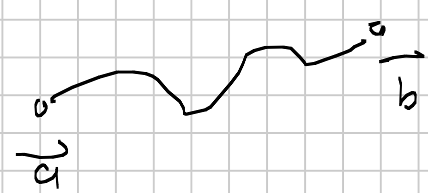
Since A is conn.
One of A_1 or A_2 is $\emptyset$.

i.e. $f(A) \subseteq S_1$ or
$f(A) \subseteq S_2$

arcwise
connected.

$S \subseteq \mathbb{R}^n$ -

& $\vec{a}, \vec{b} \in S$ - We say
$\vec{a}$ is "path-connected"
to $\vec{b}$ in S if there
is a cont. curve
from $\vec{a}$ to $\vec{b}$ in S



i.e. $\exists\, f : [0,1] \to S$

f - cont, &

$f(0) = \vec{a}$, $f(1) = \vec{b}$.

We say S is
arcwise conn. if all
pairs $a, b \in S$

are path connected.

Lemma In $\mathbb{R}$

a set is conn. iff it is pathwise conn.

Lemma $S \subseteq \mathbb{R}^n$

If S is pathwise conn. then it is conn.

pf Suppose S is pathwise conn but not conn.

Let S_1, S_2 disc. S.

$\vec{a} \in S_1$, $\vec{b} \in S_2$.

Let $f : [0,1] \to S$,

$f(0) = \vec{a}$, $f(1) = \vec{b}$,

f - cont.

By 1st lemma –

$f([0,1]) \subseteq S_1$

or $\subseteq S_2$

But it hits both

$\Rightarrow \Leftarrow$.

Example:

$V = \{(0,y),\ |y| \le 1\}$

$U = \{(x, \sin(\pi/x)) :$

$x \in (0,2]\}$

$S = U \cup V$.

S is connected –

Suppose S_1, S_2

disc. S.

either $V \subseteq S_1$ or
$V \subseteq S_2$

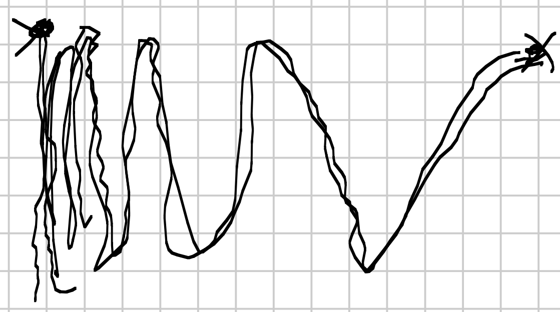
U is the cont. image of $[0,2]$,
$x \to (x, \sin \pi/x)$

U is conn.
$\Rightarrow U \subseteq S_1$
or $U \subseteq S_2$,

$S = U \cup V$.
$\Rightarrow U = S_1$, $V = S_2$
or vice versa.

But $V \subseteq \overline{U}$.

$\overline{S}_1 \cap S_2 \neq \emptyset$
so S is conn.



Not pathwise
(skip)

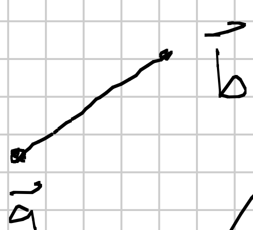
Thm If S is conn & open then it is arcwise connected.

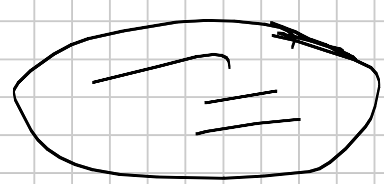
Convex Sets

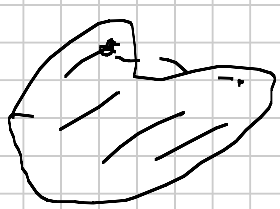
A set $A \subseteq \mathbb{R}^n$ is **convex** if whenever $\vec{a}, \vec{b} \in A$ the entire line segment

$$\{t\vec{a} + (1-t)\vec{b} : t \in [0,1]\}$$

is in A







Convex sets are always pathwise conn.

Uniform Continuity

f is cont. at x means $\forall \varepsilon, \exists \delta$ etc

If f is cont everywhere then $\forall x, \forall \varepsilon, \exists \delta(\varepsilon, x)$

δ - depends on x.

If $\delta > 0$, $\delta(\varepsilon)$ only depends on ε & not on x we say f is unif. continuous.

$f(x) = \sin x$ -

$\delta = \varepsilon$.

$f(x) = x^2$

For any $\delta > 0$

$$|(x+\delta)^2 - x^2|$$

$$= |2x\delta + \delta^2|$$

$$x > \frac{1}{\delta}$$

this is ≥ 2.

<u>Thm</u> Suppose $D \subseteq \mathbb{R}^n$ is compact & $f: D \to \mathbb{R}^m$ is cont. Then f is unif. cont.

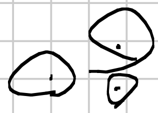
<u>pf</u> Let $\varepsilon > 0$.

For each $\vec{x} \in D$, $\exists\ \delta_{\vec{x}}$ & if

$$\|\vec{x} - \vec{y}\| < \delta_x$$

then $\|f(\vec{x}) - f(\vec{y})\| < \varepsilon/2$.

$$\mathcal{U} = \{ B_{\delta_x/2}(x) \}.$$



$\mathcal{U}$ is an open cover for D.

D - compact so $\exists$

$x_1, x_2 \dots x_k$ + $B_{\delta_{x_i}/2}(x_i)$ cover D

$$\min\left(\delta_{x_1}/2, \delta_{x_2}/2 \dots \delta_{x_k}/2\right) = \delta.$$

$$\|\vec{x} - \vec{y}\| < \delta$$

$\exists i$: $\vec{x} \in B_{\delta_{x_i}/2}(x_i)$

$$\|\vec{x} - \vec{x}_i\| < \delta_{x_i}/2 < \delta_{x_i}$$

$$\|\vec{y} - \vec{x}_i\| \leq$$

$$\|\vec{y} - \vec{x}\| + \|\vec{x} - \vec{x}_i\| < \delta_{x_i}.$$

$$\to \|f(\vec{x}) - f(\vec{x}_{i})\| < \varepsilon/2$$

$$\to \|f(\vec{y}) - f(x_{i})\| < \varepsilon/2$$

$$\|f(\vec{x}) - f(\vec{y})\| \leq$$

$$\|f(\vec{x}) - f(\vec{x}_{i})\| + \|f(\vec{y}) - f(x_{i})\|$$

$$< \varepsilon$$