Sept 2

Note Title 9/2/2009

Open sets
Closed sets

Thm If $\mathcal{A}$ is a collection of open sets. Then $\bigcup_{A \in \mathcal{A}} A$ is open.

Thm If $A_1 \dots A_k$ is a finite collection of open sets. Then $\bigcap_{i=1}^{k} A_i$ is open

BUT an inf. intersection need not be open!

pf $\vec{a} \in \bigcap_{i=1}^{k} A_i$.

$\Rightarrow \forall i, \ \vec{a} \in A_i$. A_i - open

$\Rightarrow \exists \ r_i >, \ B_{r_i}(\vec{a}) \subseteq A_i$.

Set $r = \min(r_1 \cdots r_k) > 0$

& $B_r(\vec{a}) \subseteq B_{r_i}(\vec{a}) \subseteq A_i$

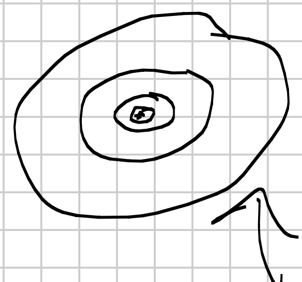
$\Rightarrow \forall i, \ B_r(\vec{a}) \subseteq A_i$

$\Rightarrow B_r(\vec{a}) \subseteq \bigcap_{i=1}^{k} A_i$

Let $r_i = 10^{-i}$

$B_{r_i}(\vec{0})$ are <u>open</u>.

$\bigcap_{i=1}^{\infty} B_{r_i}(\vec{0})$



Claim $\bigcap_{i=1}^{\infty} B_{r_i}(\vec{0}) = \{\vec{0}\}$.

① $\vec{0} \in B_{r_i}(\vec{0}) \ \forall i$.

② Suppose $\vec{a} \in \bigcap_{i=1}^{\infty} B_{r_i}(\vec{0})$, $\vec{a} \neq \vec{0}$.

$\Rightarrow \quad \|\vec{a} - \vec{0}\| > 0$

$\Rightarrow \quad \exists i, \quad \|\vec{a}\| > 10^{-i}$

$\Rightarrow \quad \vec{a} \notin B_{r_i}(\vec{0})$

$\Rightarrow \quad \vec{a} \notin \bigcap_{i=1}^{\infty} B_{r_i}(\vec{0}).$

- Closed sets -

S is closed if
$\partial S \subseteq S.$

Thm

S is closed
iff S^c is open.

pf S - closed

$\Leftrightarrow \partial S \subseteq S$

$\Leftrightarrow \partial S \cap S^c = \emptyset$

$\Leftrightarrow \partial(S^c) \cap S^c = \emptyset$

$\Leftrightarrow (S^c)^{int} = S^c$

$\iff$ $\underline{S^c\text{-open}}$

<u>def</u> $\overline{S} = S \cup \partial S$

<u>Lemma</u>

$$\overline{S} = \underline{\underline{((S^c)^{int})^c}}$$

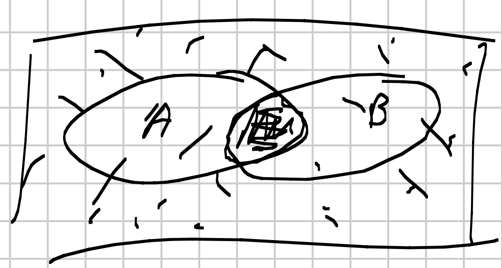
<u>pf</u>

$$(S^c)^{int} = S^c - \partial(S^c)$$

<u>so</u> $((S^c)^{int})^c$

$$\underline{S^c \cap (\partial(S^c))^c}$$

$$(S^c \cap (\partial(S^c))^c)^c$$



$$= S \cup \partial(S^c)$$

$$= S \cup \partial(S) = \overline{S}$$

Thm $\overline{\overline{S}} = \overline{S}$

pf $\overline{(\overline{S})}$

$$= \left(\left(\left(\left((S^c)^{int}\right)^{\cancel{c}}\right)^{\cancel{c}}\right)^{int}\right)^c$$

$$= \left(\left((S^c)^{int}\right)^{\cancel{int}}\right)^c$$

$$= \left((S^c)^{int}\right)^c = \overline{S}.$$

Cor S is closed iff $\overline{S} = S$.

Thm If $\mathcal{C}$ is any collection of closed sets then

$$\bigcap_{C \in \mathcal{C}} C \text{ is closed.}$$

Thm If $C_1 \cdots C_k$

is a finite collection
of closed sets then
$$\bigcup_{i=1}^{k} C_i \text{ is closed}$$

But an inf. union need
not be.

pf $\mathcal{C} \sim \mathcal{A} = \{C^c \mid C \in \mathcal{C}\}$

$\mathcal{A}$ - are open sets -

$$\bigcup_{C^c \in \mathcal{A}} C^c \text{ - open}$$

$$\left(\bigcup_{C \in \mathcal{C}} C^c\right)^c \text{ - closed}$$

$$= \bigcap_{C \in \mathcal{C}} (C^c)^c = \bigcap_{C \in \mathcal{C}} C$$

Give me an
inf list of closed
sets whose

union is not closed

$\{\{\vec{a}\}: \vec{a} \in B_1(\vec{0})\}$

Limits.

functions:

$f: D \to \mathbb{R}$,

$D \subseteq \mathbb{R}^n$.

polynomials

$x^2 + xy - 5y$

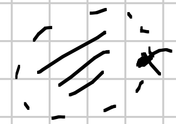
rational fctns.

$$\frac{x^2 - 5xy}{2xy - 1}$$

$2xy \neq 1$, $x \neq \frac{1}{2y}$

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = L$$

1) Need $\vec{a} \in \overline{(\mathrm{Dom} f)}$



2) $\forall \varepsilon > 0, \ \exists \delta > 0$

& if $0 < \|\vec{x} - \vec{a}\| < \delta$

& $\vec{x} \in \mathrm{Dom} f$

then

$|f(\vec{x}) - L| < \varepsilon.$

If $\vec{a} \in \mathrm{Dom} f$ &

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a}).$$

then we say f is continuous at $\vec{a}$.