Note Title 9/16/2009

Sept 16

Subsequences

a_k - sequence

Let $k : \mathbb{N} \to \mathbb{N}$

$k_{i+1} > k_i$.

then

$b_i = a_{k_i}$ is a

<u>subsequence</u>

<u>Lemma</u> If

$$\lim_{k \to \infty} a_k = L$$

& a_{k_i} is a subseq.

then $\lim_{i \to \infty} a_{k_i} = L$.

Defn We say
a_k is "increasing"
if $a_{k+1} \geq a_k$.

a_k is "decreasing"
if $a_{k+1} \leq a_k$
& "Monotone" if it
is one of these.

Lemma
a) If a_k is increasing
& bounded above then
it converges.
b) If a_k is decreasing
& bounded below then
it converges.

pf $L = \sup\{a_k\}$.
Let $\varepsilon > 0$. $\exists\, a_K$,
$$a_K \leq L < a_K + \varepsilon.$$

$k \geq K$

$$a_K \leq \underline{\underline{a_k}} \leq \underline{\underline{L}} < a_K + \varepsilon \leq \underline{\underline{a_k + \varepsilon}}$$

$$\Rightarrow \underline{\underline{|L - a_k| \leq \varepsilon}}.$$

$$\underline{a_k \to L}$$

<u>Thm</u> Every $\mathbb{R}$-valued sequence has a monotone subsequence.

pf.

Case 1. a_k is unbounded above.

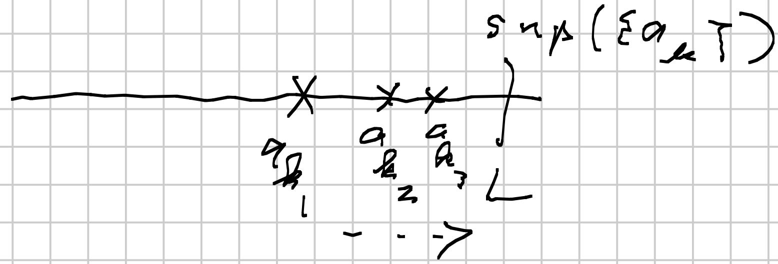
$\exists$ subs $\nearrow \infty$.

Case 2. a_k is unbounded below

$\exists$ subs $\searrow -\infty$

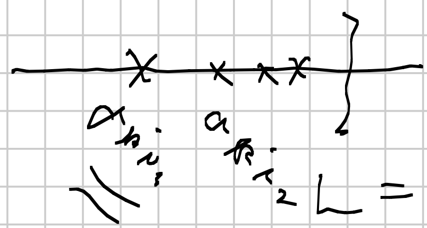
Case 3. a_k are bounded &

$\sup(\{a_k\})$ is <u>not</u> a max.



$a_{k_i} \nearrow$

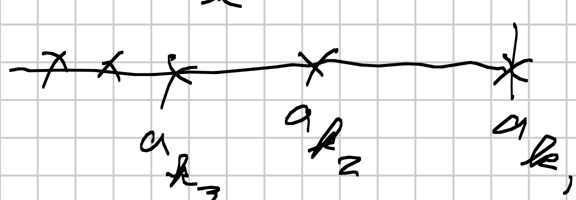
Case 4 a_k is bounded & has a subsequence a_{k_i} whose sup is not a max.



$L = \sup(\{a_{k_i}\})$

Case 5 a_k is bounded & for all subsequences, the sup is a max.

a_k



$\sup(a_{k_1+1}, a_{k_1+2} \cdots) = a_{k_2}$

$\sup(a_{k_2+1}, a_{k_2+2}, \dots) = a_{k_3}$

$\vdots$

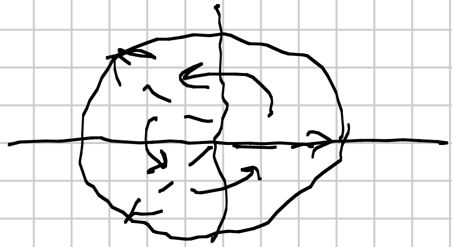
a_{k_i} <u>decrease</u>.

Cor. Every bounded sequence has a convergent subsequence.

Subsequential limit value

a_k — L is a subsequential limit value if $\exists\, k_i \ni a_{k_i} \to L$.

Ex $a_k = \sin\left(\underbrace{\frac{2\pi k}{3} + \frac{1}{k}}\right)$.



$k - k = 1 \bmod 3$
$= 2 \bmod 3$
$= 0 \bmod 3$

$k = 1 \bmod 3$

$$a_k = \sin\left(\frac{2\pi}{3} + \frac{1}{k}\right) \to \frac{\sqrt{3}}{2}$$

π

$k = 2 \bmod 3$

$$a_k = \sin\left(\frac{4\pi}{3} + \frac{1}{k}\right) \to -\frac{\sqrt{3}}{2}$$

$k = 0 \bmod 3$

$$a_k = \sin\left(\frac{1}{k}\right) \to 0.$$

Ex

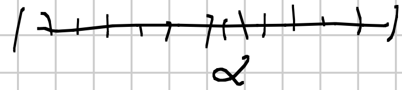
$$\underbrace{0, 1}, \underbrace{0, \frac{1}{2}, 1}, \underbrace{0, \frac{1}{3}, \frac{2}{3}, 1}, \underbrace{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1},$$

$$\ldots, \underbrace{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1}, \ldots$$

a_k "

$$B_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\right\}$$

$\alpha \in [0,1]$



$$\exists\ a_{k_n} \in B_n,$$

$$|a_{k_n} - \alpha| < \frac{1}{n}.$$

$k_{n+1} > k_n$, a_{k_n} is a subseq

$$|a_{k_n} - \alpha| < \frac{1}{n}$$

$$\lim_{n \to \infty} |a_{k_n} - \alpha| = 0$$

$$\underline{a_{k_n} \to \alpha.}$$

Hint

$[a, b]$

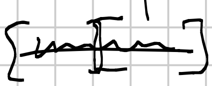
$\left[a, a + \frac{b-a}{2}\right]$ $\left[a + \frac{b-a}{2}, b\right]$

I_1 I_2

$\underline{\underline{I_1 \cap S}}$, $\underline{\underline{I_2 \cap S}}$

Assume $I_1 \cap S$ is inf.

I_1



$I_{1,1}$ $I_{1,2}$

$I_{1,1} \cap S$, $I_{1,2} \cap S$